A Brief Biography of Grassmann

Hermann Günther Grassmann was born in 1809 in Stettin, a town in Pomerania a short distance inland from the Baltic. His father Justus Günther Grassmann taught mathematics and physical science at the Stettin Gymnasium. Hermann was no child prodigy. His father used to say that he would be happy if Hermann became a craftsman or a gardener.

In 1827 Grassmann entered the University of Berlin with the intention of studying theology. As his studies progressed he became more and more interested in studying philosophy. At no time whilst a student in Berlin was he known to attend a mathematics lecture.

Grassmann was however only 23 when he made his first important geometric discovery: a method of adding and multiplying lines. This method was to become the foundation of his *Ausdehnungslehre* (extension theory). His own account of this discovery is given below.

Grassmann was interested ultimately in a university post. In order to improve his academic standing in science and mathematics he composed in 1839 a work (over 200 pages) on the study of tides entitled *Theorie der Ebbe und Flut*. This work contained the first presentation of a system of spacial analysis based on vectors including vector addition and subtraction, vector differentiation, and the elements of the linear vector function, all developed for the first time. His examiners failed to see its importance.

Around Easter of 1842 Grassmann began to turn his full energies to the composition of his first '*Ausdehnungslehre*', and by the autumn of 1843 he had finished writing it. The following is an excerpt from the foreword in which he describes how he made his seminal discovery. The translation is by Lloyd Kannenberg (Grassmann 1844).

> The initial incentive was provided by the consideration of negatives in geometry; I was used to regarding the displacements *AB* and *BA* as opposite magnitudes. From this it follows that if *A*, *B*, *C* are points of a straight line, then *AB* + *BC* = *AC* is always true, whether *AB* and *BC* are directed similarly or *oppositely*, that is even if *C* lies between *A* and *B*. In the latter case *AB* and *BC* are not interpreted merely as lengths, but rather their directions are simultaneously retained as well, according to which they are precisely oppositely oriented. Thus the distinction was drawn between the sum of lengths and the sum of such displacements in which the directions were taken into account. From this there followed the demand to establish this latter concept of a sum, not only for the case that the displacements were similarly or oppositely directed, but also for all other cases. This can most easily be accomplished if the law *AB* + *BC* = *AC* is imposed even when *A*, *B*, *C* do not lie on a single straight line.
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> Thus the first step was taken toward an analysis that subsequently led to the new branch of mathematics presented here. However, I did not then recognize the rich and fruitful domain I had reached; rather, that result seemed scarcely worthy of note until it was combined with a related idea.
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> While I was pursuing the concept of product in geometry as it had been established by my father, I concluded that not only rectangles but also parallelograms in general may be regarded as products of an adjacent pair of their sides, provided one again interprets the

> product, not as the product of their lengths, but as that of the two displacements with their directions taken into account. When I combined this concept of the product with that previously established for the sum, the most striking harmony resulted; thus whether I multiplied the sum (in the sense just given) of two displacements by a third displacement lying in the same plane, or the individual terms by the same displacement and added the products with due regard for their positive and negative values, the same result obtained, and must always obtain.
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> This harmony did indeed enable me to perceive that a completely new domain had thus been disclosed, one that could lead to important results. Yet this idea remained dormant for some time since the demands of my job led me to other tasks; also, I was initially perplexed by the remarkable result that, although the laws of ordinary multiplication, including the relation of multiplication to addition, remained valid for this new type of product, one could only interchange factors if one simultaneously changed the sign (i.e. changed + into – and vice versa).

As with his earlier work on tides, the importance of this work was ignored. Since few copies were sold, most ended by being used as waste paper by the publisher. The failure to find acceptance for Grassmann's ideas was probably due to two main reasons. The first was that Grassmann was just a simple schoolteacher, and had none of the academic charisma that other contemporaries, like Hamilton for example, had. History seems to suggest that the acceptance of radical discoveries often depends more on the discoverer than the discovery.

The second reason is that Grassmann adopted the format and the approach of the modern mathematician. He introduced and developed his mathematical structure axiomatically and abstractly. The abstract nature of the work, initially devoid of geometric or physical significance, was just too new and formal for the mathematicians of the day and they all seemed to find it too difficult. More fully than any earlier mathematician, Grassmann seems to have understood the associative, commutative and distributive laws; yet still, great mathematicians like Möbius found it unreadable, and Hamilton was led to write to De Morgan that to be able to read Grassmann he 'would have to learn to smoke'.

In the year of publication of the *Ausdehnungslehre* (1844) the Jablonowski Society of Leipzig offered a prize for the creation of a mathematical system fulfilling the idea that Leibniz had sketched in 1679. Grassmann entered with '*Die Geometrische Analyse geknüpft und die von Leibniz Characteristik*', and was awarded the prize. Yet as with the *Ausdehnungslehre* it was subsequently received with almost total silence.

However, in the few years following, three of Grassmann's contemporaries were forced to take notice of his work because of priority questions. In 1845 Saint-Venant published a paper in which he developed vector sums and products essentially identical to those already occurring in Grassmann's earlier works (Barré 1845). In 1853 Cauchy published his method of 'algebraic keys' for solving sets of linear equations (Cauchy 1853). Algebraic keys behaved identically to Grassmann's units under the exterior product. In the same year Saint-Venant published an interpretation of the algebraic keys geometrically and in terms of determinants (Barré 1853). Since such were fundamental to Grassmann's already published work he wrote a reply for Crelle's Journal in 1855 entitled '*Sur les différentes genres de multiplication*' in which he claimed priority over Cauchy and Saint-Venant and published some new results (Grassmann 1855).

It was not until 1853 that Hamilton heard of the *Ausdehnungslehre*. He set to reading it and soon after wrote to De Morgan.

> I have recently been reading … more than a hundred pages of Grassmann's *Ausdehnungslehre*, with great admiration and interest …. If I could hope to be put in rivalship with Des Cartes on the one hand and with Grassmann on the other, my scientific ambition would be fulfilled.

During the period 1844 to 1862 Grassmann published seventeen scientific papers, including important papers in physics, and a number of mathematics and language textbooks. He edited a political paper for a time and published materials on the evangelization of China. This, on top of a heavy teaching load and the raising of a large family. However, this same period saw only few mathematicians — Hamilton, Cauchy, Möbius, Saint-Venant, Bellavitis and Cremona — having any acquaintance with, or appreciation of, his work.

In 1862 Grassmann published a completely rewritten *Ausdehnungslehre*: *Die Ausdehnungslehre: Vollständing und in strenger Form*. In the foreword Grassmann discussed the poor reception accorded his earlier work and stated that the content of the new book was presented in 'the strongest mathematical form that is actually known to us; that is Euclidean …'. It was a book of theorems and proofs largely unsupported by physical example.

This apparently was a mistake, for the reception accorded this new work was as quiet as that accorded the first, although it contained many new results including a solution to Pfaff's problem. Friedrich Engel (the editor of Grassmann's collected works) comments: 'As in the first *Ausdehnungslehre* so in the second: matters which Grassmann had published in it were later independently rediscovered by others, and only much later was it realized that Grassmann had published them earlier' (Engel 1896).

Thus Grassmann's works were almost totally neglected for forty-five years after his first discovery. In the second half of the 1860s recognition slowly started to dawn on his contemporaries, among them Hankel, Clebsch, Schlegel, Klein, Noth, Sylvester, Clifford and Gibbs. Gibbs discovered Grassmann's works in 1877 (the year of Grassmann's death), and Clifford discovered them in depth about the same time. Both became quite enthusiastic about Grassmann's new mathematics.

Grassmann's activities after 1862 continued to be many and diverse. His contribution to philology rivals his contribution to mathematics. In 1849 he had begun a study of Sanskrit and in 1870 published his *Wörtebuch zum Rig-Veda*, a work of 1784 pages, and his translation of the Rig-Veda, a work of 1123 pages, both still in use today. In addition he published on mathematics, languages, botany, music and religion. In 1876 he was made a member of the American Oriental Society, and received an honorary doctorate from the University of Tübingen.

On 26 September 1877 Hermann Grassmann died, departing from a world only just beginning to recognize the brilliance of the mathematical creations of one of its most outstanding eclectics.