Bibliography

To be completed

Armstrong H L 1959

'On an alternative definition of the vector product in n-dimensional vector analysis'
Matrix and Tensor Quarterly, **IX** no 4, pp 107-110.

The author proposes a definition equivalent to the complement of the exterior product of n–1 vectors.

Ball R S 1900

A Treatise on the Theory of Screws
Cambridge University Press (1900). Reprinted 1998.

A classical work on the theory of screws, containing an annotated bibliography in which Ball refers to the *Ausdehnungslehre* of 1862: 'This remarkable work … contains much that is of instruction and interest in connection with the present theory … . Here we have a very general theory which includes screw coordinates as a special case.' Ball does not use Grassmann's methods in his treatise.

Barton H 1927

'A Modern Presentation of Grassmann's Tensor Analysis'
American Journal of Mathematics, **XLIX**, pp 598-614.

This paper covers similar ground to that of Moore (1926).

Bowen R M and Wang C-C 1976

Introduction to Vectors and Tensors
Plenum Press. Two volumes.

This is the only contemporary text on vectors and tensors I have sighted which relates points and vectors via the explicit introduction of the origin into the calculus (p 254).

Brand L 1947

Vector and Tensor Analysis
Wiley, New York.

Contains a chapter on motor algebra which, according to the author in his preface '… is apparently destined to play an important role in mechanics as well as in line geometry'. There is also a chapter on quaternions.

Buchheim A 1884-1886

'On the Theory of Screws in Elliptic Space'
Proceedings of the London Mathematical Society, **xiv** (1884) pp 83-98, **xvi** (1885) pp 15-27, **xvii** (1886) pp 240-254, **xvii** p 88.

The author writes 'My special object is to show that the *Ausdehnungslehre* supplies all the necessary materials for a calculus of screws in elliptic space. Clifford was apparently led to construct his theory of biquaternions by the want of such a calculus, but Grassmann's method seems to afford a simpler and more natural means of expression than biquaternions.' (*xiv*, p 90) Later he extends this theory to '… all kinds of space.' (*xvi*, p 15)

Burali-Forti C 1897

Introduction à la Géométrie Différentielle suivant la Méthode de H. Grassmann
Gauthier-Villars, Paris

This work covers both algebra and differential geometry in the tradition of the Peano approach to the *Ausdehnungslehre*.

Burali-Forti C and Marcolongo R 1910

Éléments de Calcul Vectoriel
Hermann, Paris

Mostly a treatise on standard vector analysis, but it does contain an appendix (pp 176-198) on the methods of the *Ausdehnungslehre* and some interesting historical notes on the vector calculi and their notations. The authors use the wedge $\wedge$ to denote Gibbs' cross product $\times$ and use the $\times$, initially introduced by Grassmann to denote the scalar or inner product.

Carvallo M E 1892

'La Méthode de Grassmann'
Nouvelles Annales de Mathématiques, serie 3 **XI**, pp 8-37.

An exposition of some of Grassmann's methods applied to three dimensional geometry following the approach of Peano. It does not treat the interior product.

Chevalley C 1955

The Construction and Study of Certain Important Algebras
Mathematical Society of Japan, Tokyo.

Lectures given at the University of Tokyo on graded, tensor, Clifford and exterior algebras.

Clifford W K 1873

'Preliminary Sketch of Biquaternions'
Proceedings of the London Mathematical Society, **IV**, nos 64 and 65, pp 381-395

This paper includes an interesting discussion of the geometric nature of mechanical quantities. Clifford adopts the term 'rotor' for the bound vector, and 'motor' for the general sum of rotors. By analogy with the quaternion as a quotient of vectors he defines the biquaternion as a quotient of motors.

Clifford W K 1878

'Applications of Grassmann's Extensive Algebra'
American Journal of Mathematics Pure and Applied, **I**, pp 350-358

In this paper Clifford lays the foundations for general Clifford algebras.

Clifford W K 1882

Mathematical Papers
Reprinted by Chelsea Publishing Co, New York (1968).

Of particular interest in addition to his two published papers above are the otherwise unpublished notes:
'Notes on Biquaternions' (~1873)
'Further Note on Biquaternions' (1876)
'On the Classification of Geometric Algebras' (1876)
'On the Theory of Screws in a Space of Constant Positive Curvature' (1876).

Coffin J G 1909

Vector Analysis
Wiley, New York.

This is the second English text in the Gibbs-Heaviside tradition. It contains an appendix comparing the various notations in use at the time, including his view of the Grassmannian notation.

Collins J V 1899-1900

'An elementary Exposition of Grassmann's *Ausdehnungslehre* or Theory of Extension'
American Mathematical Monthly, **6** (1899) pp 193-198, 261-266, 297-301; **7** (1900) pp 31-35, 163-166, 181-187, 207-214, 253-258.

This work follows in summary form the *Ausdehnungslehre* of 1862 as regards general theory but differs in its discussion of applications. It includes applications to geometry and brief applications to linear equations, mechanics and logic.

Coolidge J L 1940

'Grassmann's Calculus of Extension' in
A History of Geometrical Methods
Oxford University Press, pp 252-257

This brief treatment of Grassmann's work is characterized by its lack of clarity. The author variously describes an exterior product as 'essentially a matrix' and as 'a vector perpendicular to the factors' (p 254). And confusion arises between Grassmann's matrix and the division of two exterior products (p 256).

Cox H 1882

'On the application of Quaternions and Grassmann's *Ausdehnungslehre* to different kinds of Uniform Space'
Cambridge Philosophical Transactions, **XIII** part II, pp 69-143.

The author shows that the exterior product is the multiplication required to describe non-metric geometry, for 'it involves no ideas of distance' (p 115). He then discusses exterior, regressive and interior products, applying them to geometry, systems of forces, and linear complexes – using the notation of 1844. In other papers Cox applies the *Ausdehnungslehre* to non-Euclidean geometry (1873) and to the properties of circles (1890).

Crowe M J 1967

A History of Vector Analysis
Notre Dame.

This is the most informative work available on the history of vector analysis from the discovery of the geometric representation of complex numbers to the development of the Gibbs-Heaviside system Crowe's thesis is that the Gibbs-Heaviside system grew mostly out of quaternions rather than from the *Ausdehnungslehre*. His explanation of Grassmannian concepts is particularly accurate in contradistinction to many who supply a more casual reference.

Dibag I 1974

'Factorization in Exterior Algebras'
Journal of Algebra, **30**, pp 259-262

The author develops necessary and sufficient conditions for an *m*-element to have a certain number of 1-element factors. He also shows that an (*n*–2)-element in an odd dimensional space always has a 1-element factor.

Grassmann H G 1878

'Verwendung der Ausdehnungslehre fur die allgemeine Theorie der Polaren und den Zusammenhang algebraischer Gebilde'
Crelle's Journal, **84**, pp 273–283.

This is Grassmann's last paper. It contains, among other material, his most complete discussion on the notion of 'simplicity'.

Gibbs J W 1886

'On multiple algebra'
Address to the American Association for the Advancement of Science
In *Collected Works*, Gibbs 1928, vol 2.

This paper is probably the most authoritative historical comparison of the different 'vectorial' algebras of the time. Gibbs was obviously very enthusiastic about the *Ausdehnungslehre*, and shows himself here to be one of Grassmann's greatest proponents.

Gibbs J W 1891

'Quaternions and the Ausdehnungslehre'
Nature, **44**, pp 79–82. Also in *Collected Works*, Gibbs 1928.

Gibbs compares Hamilton's Quaternions with Grassmann's *Ausdehnungslehre* and concludes that '... Grassmann's system is of indefinitely greater extension ...'. Here he also concludes that to Grassmann must be attributed the discovery of matrices. Gibbs published a further three papers in *Nature* (also in *Collected Works*, Gibbs 1928) on the relationship between quaternions and vector analysis, providing an enlightening insight into the quaternion–vector analysis controversy of the time.

Gibbs J W 1928

The Collected Works of J. Willard Gibbs Ph.D. LL.D.
Two volumes. Longmans, New York.

In part 2 of Volume 2 is reprinted Gibbs' only personal work on vector analysis: *Elements of Vector Analysis, Arranged for the Use of Students of Physics* (1881–1884). This was not published elsewhere.

To be completed.

A note on sources to Grassmann's work

The best source for Grassmann's contributions to science is his *Collected Works* (Grassmann 1896) which contain in volume 1 both *Die Ausdehnungslehre von 1844* and *Die Ausdehnungslehre von 1862*, as well as *Geometrische Analyse*, his prizewinning essay fulfilling Leibniz's search for an algebra of geometry. Volume 2 contains papers on geometry, analysis, mechanics and physics, while volume 3 contains *Theorie der Ebbe und Flut*.

Die Ausdehnungslehre von 1862, fully titled: *Die Ausdehnungslehre. Vollständig und in strenger Form* is perhaps Grassmann's most important mathematical work. It comprises two main parts: the first devoted basically to the *Ausdehnungslehre* (212 pages) and the second to the theory of functions (155 pages). The Collected Works edition contains 98 pages of notes and comments. The discussion on the *Ausdehnungslehre* includes chapters on addition and subtraction, products in general, progressive and regressive products, interior products, and applications to geometry. A Cartesian metric is assumed.

Both Grassmann's *Ausdehnungslehre* have been translated into English by Lloyd C Kannenberg. The 1844 version is published as *A New Branch of Mathematics: The Ausdehnungslehre of 1844 and Other Works,* Open Court 1995. The translation contains *Die Ausdehnungslehre von 1844*, *Geometrische Analyse*, selected papers on mathematics and physics, a bibliography of Grassmann's principal works, and extensive editorial notes. The 1862 version is published as *Extension Theory*. It contains work on both the theory of extension and the theory of functions. Particularly useful are the editorial and supplementary notes.

Apart from these translations, probably the best and most complete exposition on the *Ausdehnungslehre* in English is in Alfred North Whitehead's *A Treatise on Universal Algebra* (Whitehead 1898). Whitehead saw Grassmann's work as one of the foundation stones on which he hoped to build an algebraic theory which united the several important and new mathematical systems which emerged during the nineteenth century — the algebra of symbolic logic, Grassmann's theory of extension, quaternions, matrices and the general theory of linear algebras.

The second most complete exposition of the *Ausdehnungslehre* is Henry George Forder's *The Theory of Extension* (Forder 1941). Forder's interest is mainly in the geometric applications of the theory of extension.

The only other books on Grassmann in English are those by Edward Wyllys Hyde, *The Directional Calculus* (Hyde 1890) and *Grassmann's Space Analysis* (Hyde 1906). They treat the theory of extension in two and three-dimensional geometric contexts and include some applications to statics. Several topics such as Hyde's treatment of screws are original contributions.

The seminal papers on Clifford algebra are by William Kingdon Clifford and can be found in his collected works *Mathematical Papers* (Clifford 1882), republished in a facsimile edition by Chelsea.

Fortunately for those interested in the evolution of the emerging 'geometric algebras', *The International Association for Promoting the Study of Quaternions and Allied Systems of Mathematics* published a bibliography (Macfarlane 1913) which, together with supplements to 1913, contains about 2500 articles. This therefore most likely contains all the works on the *Ausdehnungslehre* and related subjects up to 1913.

The only other recent text devoted specifically to Grassmann algebra (to the author's knowledge as of 2001) is Arno Zaddach's *Grassmanns Algebra in der Geometrie*, BI-Wissenschaftsverlag, (Zaddach 1994).