

11 Exploring Hypercomplex Algebra



11.1 Introduction

11.2 Some Initial Definitions and Properties

- The conjugate
- Distributivity
- The norm
- Factorization of scalars
- Multiplication by scalars
- The real numbers \mathbb{R}

11.3 The Complex Numbers \mathbb{C}

- Constraints on the hypercomplex signs
- Complex numbers as Grassmann numbers under the hypercomplex product

11.4 The Hypercomplex Product in a 2-Space

- ⊗ Tabulating products in 2-space
- The hypercomplex product of two 1-elements
- The hypercomplex product of a 1-element and a 2-element
- The hypercomplex square of a 2-element
- The product table in terms of exterior and interior products

11.5 The Quaternions \mathbb{Q}

- The product table for orthonormal elements
- Generating the quaternions
- The norm of a quaternion
- The Cayley-Dickson algebra

11.6 The Norm of a Grassmann number

- The norm
- The norm of a simple m -element
- The skew-symmetry of products of elements of different grade
- The norm of a Grassmann number in terms of hypercomplex products
- The norm of a Grassmann number of simple components
- The norm of a non-simple element

11.7 Products of two different elements of the same grade

- The symmetrized sum of two m -elements
- Symmetrized sums for elements of different grades
- The body of a symmetrized sum

The soul of a symmetrized sum
Summary of results of this section

11.8 Octonions

To be completed

11.1 Introduction

In this chapter we explore the relationship of Grassmann algebra to the hypercomplex algebras. Typical examples of hypercomplex algebras are the algebras of the real numbers \mathbb{R} , complex numbers \mathbb{C} , quaternions \mathbb{Q} , octonions (or Cayley numbers) \mathbb{O} and Clifford algebras C . We will show that each of these algebras can be generated as an algebra of Grassmann numbers under a new product which we take the liberty of calling the *hypercomplex product*.

This hypercomplex product of an m -element α and a k -element β is denoted $\alpha \circ \beta$ and defined as a sum of signed generalized products of α with β .

$$\alpha \circ \beta = \sum_{\lambda=0}^{\text{Min}[m,k]} \sigma^{\mathbf{m},\lambda,\mathbf{k}} \alpha \Delta_{\mathbf{m}\lambda\mathbf{k}} \beta \quad 11.1$$

Here, the sign $\sigma^{\mathbf{m},\lambda,\mathbf{k}}$ is a function of the grades \mathbf{m} and \mathbf{k} of the factors and the order λ of the generalized product and takes the values +1 or -1 depending on the type of hypercomplex product being defined.

Note particularly that this is an 'invariant' approach, not requiring the algebra to be defined in terms of generators or basis elements. We will see in what follows that this leads to considerably more insight on the nature of the hypercomplex numbers than afforded by the usual approach. Because the generalized Grassmann products on the right-hand side may represent the only geometrically valid constructs from the factors on the left, it is enticing to postulate that all algebras with a geometric interpretation can be defined by such a product of Grassmann numbers.

It turns out that the real numbers are hypercomplex numbers in a space of zero dimensions, the complex numbers are hypercomplex numbers in a space of one dimension, the quaternions are hypercomplex numbers in a space of two dimensions, and the octonions are hypercomplex numbers in a space of three dimensions.

In Chapter 12: Exploring Clifford Algebra we will also show that the Clifford product may be defined in a space of any number of dimensions as a hypercomplex product with signs:

$$\sigma^{\mathbf{m},\lambda,\mathbf{k}} = (-1)^{\lambda(\mathbf{m}-\lambda) + \frac{1}{2}\lambda(\lambda-1)}$$

We could also explore the algebras generated by products of the type defined by definition 11.1, but which have some of the $\overset{m,\lambda,k}{\sigma}$ defined to be zero. For example the relations,

$$\overset{m,\lambda,k}{\sigma} = 1, \lambda = 0 \qquad \overset{m,\lambda,k}{\sigma} = 0, \lambda > 0$$

defines the algebra with the hypercomplex product reducing to the exterior product, and

$$\overset{m,\lambda,k}{\sigma} = 1, \lambda = \text{Min}[m, k] \qquad \overset{m,\lambda,k}{\sigma} = 0, \lambda \neq \text{Min}[m, k]$$

defines the algebra with the hypercomplex product reducing to the interior product.

Both of these definitions however lead to products having zero divisors, that is, some products can be zero even though neither of its factors is zero. Because one of the principal characteristics of the hypercomplex product is that it has no zero divisors, we shall limit ourselves in this chapter to exploring just those algebras. That is, we shall always assume that $\overset{m,\lambda,k}{\sigma}$ is not zero.

We begin by exploring the implications of definition 11.1 to the generation of various hypercomplex algebras by considering spaces of increasing dimension, starting with a space of zero dimensions generating the real numbers. In order to embed the algebras defined on lower dimensional spaces within those defined on higher dimensional spaces we will maintain the lower dimensional relations we determine for the $\overset{m,\lambda,k}{\sigma}$ when we explore the higher dimensional spaces.

Throughout the development we will see that the various hypercomplex algebras could be developed in alternative fashions by various combinations of signs $\overset{m,\lambda,k}{\sigma}$ and metrics. However, we shall for simplicity generally restrict the discussion to constraining the signs only and keeping products of the form $\mathbf{e}_i \ominus \mathbf{e}_i$ positive.

Although in this chapter we develop the hypercomplex product by determining more and more constraints on the signs $\overset{m,\lambda,k}{\sigma}$ that give us the properties we want of the number systems \mathbb{R} , \mathbb{C} , \mathbb{Q} , and \mathbb{O} , it can be seen that this is only one direction out of the multitude possible. Many other consistent algebras could be developed by adopting other constraints amongst the signs $\overset{m,\lambda,k}{\sigma}$.

$$\mathbf{e}_i \ominus \mathbf{e}_i$$

11.2 Some Initial Definitions and Properties

The conjugate

The conjugate of a Grassmann number \mathbf{X} is denoted \mathbf{X}^c and is defined as the body \mathbf{X}_b (scalar part) of \mathbf{X} minus the soul \mathbf{X}_s (non-scalar part) of \mathbf{X} .

$$\mathbf{X} = \mathbf{X}_b + \mathbf{X}_s \quad \mathbf{X}^c = \mathbf{X}_b - \mathbf{X}_s \quad 11.2$$

Distributivity

The first property we define of the Hypercomplex product is its distributivity.

$$\left(\sum_i \alpha_{m_i} \right) \circ_k \beta = \sum_i \left(\alpha_{m_i} \circ_k \beta \right) \quad 11.3$$

$$\beta \circ_k \left(\sum_i \alpha_{m_i} \right) = \sum_i \left(\beta \circ_k \alpha_{m_i} \right) \quad 11.4$$

The norm

The norm of a Grassmann number \mathbf{X} is denoted $n[\mathbf{X}]$ and is defined as the hypercomplex product of \mathbf{X} with its conjugate.

$$n[\mathbf{X}] = \mathbf{X} \circ \mathbf{X}^c = (\mathbf{X}_b + \mathbf{X}_s) \circ (\mathbf{X}_b - \mathbf{X}_s) = \mathbf{X}_b^2 - \mathbf{X}_s \circ \mathbf{X}_s \quad 11.5$$

Factorization of scalars

From the definition of the hypercomplex product we can use the properties of the generalized Grassmann product to show that scalars may be factorized out of any hypercomplex products:

$$\left(\mathbf{a} \alpha_{m_i} \right) \circ_k \beta = \alpha_{m_i} \circ_k \left(\mathbf{a} \beta \right) = \mathbf{a} \left(\alpha_{m_i} \circ_k \beta \right) \quad 11.6$$

Multiplication by scalars

The next property we will require of the hypercomplex product is that it behaves as expected when one of the elements is a scalar. That is:

$$\alpha_m \circ \mathbf{b} == \mathbf{b} \circ \alpha_m == \mathbf{b} \alpha_m \quad 11.7$$

From the relations 11.7 and the definition 11.1 we can determine the constraints on the ${}^{m,\lambda,k}\sigma$ which will accomplish this.

$$\alpha_m \circ \mathbf{b} == \sum_{\lambda=0}^{\text{Min}[m,0]} {}^{m,\lambda,0}\sigma \alpha_m \Delta_{\lambda} \mathbf{b} == {}^{m,0,0}\sigma \alpha_m \wedge \mathbf{b} == {}^{m,0,0}\sigma \mathbf{b} \alpha_m$$

$$\mathbf{b} \circ \alpha_m == \sum_{\lambda=0}^{\text{Min}[0,m]} {}^{0,\lambda,m}\sigma \mathbf{b} \Delta_{\lambda} \alpha_m == {}^{0,0,m}\sigma \mathbf{b} \wedge \alpha_m == {}^{0,0,m}\sigma \mathbf{b} \alpha_m$$

Hence the first constraints we impose on the ${}^{m,\lambda,k}\sigma$ to ensure the properties we require for multiplication by scalars are that:

$${}^{m,0,0}\sigma == {}^{0,0,m}\sigma == 1 \quad 11.8$$

The real numbers \mathbb{R}

The real numbers are a simple consequence of the relations determined above for scalar multiplication. When the grades of both the factors are zero we have that ${}^{0,0,0}\sigma == 1$. Hence:

$$\mathbf{a} \circ \mathbf{b} == \mathbf{b} \circ \mathbf{a} == \mathbf{a} \mathbf{b} \quad 11.9$$

The hypercomplex product in a space of zero dimensions is therefore equivalent to the (usual) real field product of the underlying linear space. Hypercomplex numbers in a space of zero dimensions are therefore (isomorphic to) the real numbers.

11.3 The Complex Numbers \mathbb{C}

Constraints on the hypercomplex signs

Complex numbers may be viewed as hypercomplex numbers in a space of one dimension.

In one dimension all 1-elements are of the form $\mathbf{a}\mathbf{e}$, where \mathbf{a} is a scalar and \mathbf{e} is the basis element.

Let α be a 1-element. From the definition of the hypercomplex product we see that the hypercomplex product of a 1-element with itself is the (possibly signed) scalar product.

$$\alpha \circ \alpha = \sum_{\lambda=0}^{\text{Min}[1,1]} {}^{1,\lambda,1}_{\sigma} \alpha \Delta_{\lambda} \alpha = {}^{1,0,1}_{\sigma} \alpha \wedge \alpha + {}^{1,1,1}_{\sigma} \alpha \Theta \alpha = {}^{1,1,1}_{\sigma} \alpha \Theta \alpha$$

In the usual notation, the product of two complex numbers would be written:

$$(a + \mathbf{i} b) (c + \mathbf{i} d) = (ac - bd) + \mathbf{i} (bc + ad)$$

In the general hypercomplex notation we have:

$$(a + b e) \circ (c + d e) = a \circ c + a \circ (d e) + (b e) \circ c + (b e) \circ (d e)$$

Simplifying this using the relations 11.6 and 11.7 above gives:

$$(ac + b d e \circ e) + (bc + ad) e = \\ \left(ac + b d {}^{1,1,1}_{\sigma} e \Theta e \right) + (bc + ad) e$$

Isomorphism with the complex numbers is then obtained by constraining ${}^{1,1,1}_{\sigma} e \Theta e$ to be -1 .

One immediate interpretation that we can explore to satisfy this is that e is a unit 1-element (with $e \Theta e = 1$), and ${}^{1,1,1}_{\sigma} = -1$.

This then is the constraint we will impose to allow the incorporation of complex numbers in the hypercomplex structure.

$${}^{1,1,1}_{\sigma} = -1$$

11.10

Complex numbers as Grassmann numbers under the hypercomplex product

The imaginary unit \mathbf{i} is then equivalent to a unit basis element (e , say) of the 1-space under the hypercomplex product.

$$\mathbf{i} = e \quad e \Theta e = 1$$

11.11

$$\mathbf{i} \circ \mathbf{i} = -\mathbf{i} \Theta \mathbf{i} = -1$$

11.12

In this interpretation, instead of the \mathbf{i} being a new entity with the special property $\mathbf{i}^2 = -1$, the focus is shifted to interpret it as a unit 1-element acting under a new

product operation \circ . Complex numbers are then interpreted as Grassmann numbers in a space of one dimension under the hypercomplex product operation.

For example, the norm of a complex number $\mathbf{a} + \beta \mathbf{i} = \mathbf{a} + \mathbf{b} \mathbf{i}$ can be calculated as:

$$n[\mathbf{a} + \beta] = (\mathbf{a} + \beta) \circ (\mathbf{a} - \beta) = \mathbf{a}^2 - \beta \circ \beta = \mathbf{a}^2 + \beta \Theta \beta = \mathbf{a}^2 + \mathbf{b}^2$$

$$n[\mathbf{a} + \beta] = \mathbf{a}^2 + \beta \Theta \beta = \mathbf{a}^2 + \mathbf{b}^2$$

11.13

11.4 The Hypercomplex Product in a 2-Space

Tabulating products in 2-space

From our previous interpretation of complex numbers as Grassmann numbers in a space of one dimension under the hypercomplex product operation, we shall require any one dimensional subspace of the 2-space under consideration to have the same properties.

The product table for the types of hypercomplex products that can be encountered in a 2-space can be entered by using the *GrassmannAlgebra* function `HypercomplexProductTable`.

```
T = HypercomplexProductTable[{1,  $\alpha$ ,  $\beta$ ,  $\alpha \wedge \beta$ }]
```

```
{ {1  $\circ$  1, 1  $\circ$   $\alpha$ , 1  $\circ$   $\beta$ , 1  $\circ$  ( $\alpha \wedge \beta$ ) },  
  { $\alpha$   $\circ$  1,  $\alpha$   $\circ$   $\alpha$ ,  $\alpha$   $\circ$   $\beta$ ,  $\alpha$   $\circ$  ( $\alpha \wedge \beta$ ) }, { $\beta$   $\circ$  1,  $\beta$   $\circ$   $\alpha$ ,  $\beta$   $\circ$   $\beta$ ,  $\beta$   $\circ$  ( $\alpha \wedge \beta$ ) },  
  {( $\alpha \wedge \beta$ )  $\circ$  1, ( $\alpha \wedge \beta$ )  $\circ$   $\alpha$ , ( $\alpha \wedge \beta$ )  $\circ$   $\beta$ , ( $\alpha \wedge \beta$ )  $\circ$  ( $\alpha \wedge \beta$ ) } }
```

To make this easier to read we format this with the *GrassmannAlgebra* function `TableTemplate`.

```
TableTemplate[T]
```

$1 \circ 1$	$1 \circ \alpha$	$1 \circ \beta$	$1 \circ (\alpha \wedge \beta)$
$\alpha \circ 1$	$\alpha \circ \alpha$	$\alpha \circ \beta$	$\alpha \circ (\alpha \wedge \beta)$
$\beta \circ 1$	$\beta \circ \alpha$	$\beta \circ \beta$	$\beta \circ (\alpha \wedge \beta)$
$(\alpha \wedge \beta) \circ 1$	$(\alpha \wedge \beta) \circ \alpha$	$(\alpha \wedge \beta) \circ \beta$	$(\alpha \wedge \beta) \circ (\alpha \wedge \beta)$

Products where one of the factors is a scalar have already been discussed. Products of a 1-element with itself have been discussed in the section on complex numbers. Applying these results to the table above enables us to simplify it to:

1	α	β	$\alpha \wedge \beta$
α	$-(\alpha \Theta \alpha)$	$\alpha \circ \beta$	$\alpha \circ (\alpha \wedge \beta)$
β	$\beta \circ \alpha$	$-(\beta \Theta \beta)$	$\beta \circ (\alpha \wedge \beta)$
$\alpha \wedge \beta$	$(\alpha \wedge \beta) \circ \alpha$	$(\alpha \wedge \beta) \circ \beta$	$(\alpha \wedge \beta) \circ (\alpha \wedge \beta)$

In this table there are four essentially different new products which we have not yet discussed.

$$\alpha \circ \beta \quad \alpha \circ (\alpha \wedge \beta) \quad (\alpha \wedge \beta) \circ \alpha \quad (\alpha \wedge \beta) \circ (\alpha \wedge \beta)$$

In the next three subsections we will take each of these products and, with a view to developing the quaternion algebra, show how they may be expressed in terms of exterior and interior products.

The hypercomplex product of two 1-elements

From the definition, and the constraint ${}^{1,1,1}\sigma = -1$ derived above from the discussion on complex numbers we can write the hypercomplex product of two (possibly) distinct 1-elements as

$$\alpha \circ \beta = {}^{1,0,1}\sigma \alpha \wedge \beta + {}^{1,1,1}\sigma (\alpha \Theta \beta) = {}^{1,0,1}\sigma \alpha \wedge \beta - (\alpha \Theta \beta)$$

The hypercomplex product $\beta \circ \alpha$ can be obtained by reversing the sign of the exterior product, since the scalar product is symmetric.

$$\begin{aligned} \alpha \circ \beta &= {}^{1,0,1}\sigma \alpha \wedge \beta - (\alpha \Theta \beta) \\ \beta \circ \alpha &= -{}^{1,0,1}\sigma \alpha \wedge \beta - (\alpha \Theta \beta) \end{aligned}$$

11.14

The hypercomplex product of a 1-element and a 2-element

From the definition of the hypercomplex product 11.1 we can obtain expressions for the hypercomplex products of a 1-element and a 2-element in a 2-space. Since the space is only of two dimensions, the 2-element may be represented without loss of generality as a product which incorporates the 1-element as one of its factors.

$$\begin{aligned} \alpha \circ (\alpha \wedge \beta) &= \sum_{\lambda=0}^{\text{Min}[1,2]} {}^{1,\lambda,2}\sigma \alpha \Delta_{\lambda} (\alpha \wedge \beta) = \\ & {}^{1,0,2}\sigma \alpha \wedge (\alpha \wedge \beta) + {}^{1,1,2}\sigma (\alpha \wedge \beta) \Theta \alpha \end{aligned}$$

$$(\alpha \wedge \beta) \circ \alpha = \sum_{\lambda=0}^{\text{Min}[2,1]} {}^{2,\lambda,1} \sigma (\alpha \wedge \beta) \Delta_{\lambda} \alpha =$$

$${}^{2,0,1} \sigma (\alpha \wedge \beta) \wedge \alpha + {}^{2,1,1} \sigma (\alpha \wedge \beta) \Theta \alpha$$

Since the first term involving only exterior products is zero, we obtain:

$$\alpha \circ (\alpha \wedge \beta) = {}^{1,1,2} \sigma (\alpha \wedge \beta) \Theta \alpha$$

$$(\alpha \wedge \beta) \circ \alpha = {}^{2,1,1} \sigma (\alpha \wedge \beta) \Theta \alpha$$
11.15

The hypercomplex square of a 2-element

All 2-elements in a space of 2 dimensions differ by only a scalar factor. We need only therefore consider the hypercomplex product of a 2-element with itself.

The generalized product of two identical elements is zero in all cases except for that which reduces to the inner product. From this fact and the definition of the hypercomplex product we obtain

$$(\alpha \wedge \beta) \circ (\alpha \wedge \beta) =$$

$$\sum_{\lambda=0}^{\text{Min}[2,2]} {}^{2,\lambda,2} \sigma (\alpha \wedge \beta) \Delta_{\lambda} (\alpha \wedge \beta) = {}^{2,2,2} \sigma (\alpha \wedge \beta) \Theta (\alpha \wedge \beta)$$

$$(\alpha \wedge \beta) \circ (\alpha \wedge \beta) = {}^{2,2,2} \sigma (\alpha \wedge \beta) \Theta (\alpha \wedge \beta)$$
11.16

The product table in terms of exterior and interior products

Collecting together the results above, we can rewrite the hypercomplex product table solely in terms of exterior and interior products and some (as yet) undetermined signs.

	α	β	α ∧ β
	-(αΘα)	${}^{1,0,1} \sigma \alpha \wedge \beta - (\alpha \Theta \beta)$	${}^{1,1,2} \sigma (\alpha \wedge \beta) \Theta$
	${}^{1,0,1} \sigma \alpha \wedge \beta - (\alpha \Theta \beta)$	-(βΘβ)	${}^{1,1,2} \sigma (\alpha \wedge \beta) \Theta$
	${}^{2,1,1} \sigma (\alpha \wedge \beta) \Theta \alpha$	${}^{2,1,1} \sigma (\alpha \wedge \beta) \Theta \beta$	${}^{2,2,2} \sigma (\alpha \wedge \beta) \Theta (\alpha$

11.17

11.5 The Quaternions \mathbb{Q}

The product table for orthonormal elements

Suppose now that α and β are orthonormal. Then:

$$\alpha \Theta \alpha = \beta \Theta \beta = 1$$

$$\begin{aligned} \alpha \Theta \beta &= 0 & (\alpha \wedge \beta) \Theta \alpha &= \\ (\alpha \Theta \alpha) \beta & & (\alpha \wedge \beta) \Theta (\alpha \wedge \beta) &= (\alpha \Theta \alpha) (\beta \Theta \beta) \end{aligned}$$

The hypercomplex product table then becomes:

1	α	β	$\alpha \wedge \beta$
α	-1	${}^{1,0,1}_{\sigma} \alpha \wedge \beta$	${}^{1,1,2}_{\sigma} \beta$
β	$-{}^{1,0,1}_{\sigma} \alpha \wedge \beta$	-1	$-{}^{1,1,2}_{\sigma} \alpha$
$\alpha \wedge \beta$	${}^{2,1,1}_{\sigma} \beta$	$-{}^{2,1,1}_{\sigma} \alpha$	${}^{2,2,2}_{\sigma}$

Generating the quaternions

Exploring a possible isomorphism with the quaternions leads us to the correspondence:

$\alpha \rightarrow \mathbf{i}$	$\beta \rightarrow \mathbf{j}$	$\alpha \wedge \beta \rightarrow \mathbf{k}$		11.18
---------------------------------	--------------------------------	--	--	-------

In terms of the quaternion units, the product table becomes:

1	\mathbf{i}	\mathbf{j}	\mathbf{k}
\mathbf{i}	-1	${}^{1,0,1}_{\sigma} \mathbf{k}$	${}^{1,1,2}_{\sigma} \mathbf{j}$
\mathbf{j}	$-{}^{1,0,1}_{\sigma} \mathbf{k}$	-1	$-{}^{1,1,2}_{\sigma} \mathbf{i}$
\mathbf{k}	${}^{2,1,1}_{\sigma} \mathbf{j}$	$-{}^{2,1,1}_{\sigma} \mathbf{i}$	${}^{2,2,2}_{\sigma}$

To obtain the product table for the quaternions we therefore require:

${}^{1,0,1}_{\sigma} = 1$	${}^{1,1,2}_{\sigma} = -1$	${}^{2,1,1}_{\sigma} = 1$	${}^{2,2,2}_{\sigma} = -1$		11.19
---------------------------	----------------------------	---------------------------	----------------------------	--	-------

These values give the quaternion table as expected.

1	i	j	k
i	-1	k	-j
j	-k	-1	i
k	j	-i	-1

11.20

Substituting these values back into the original table 11.17 gives a hypercomplex product table in terms only of exterior and interior products. This table defines the real, complex, and quaternion product operations.

1	α	β	$\alpha \wedge \beta$
α	$-(\alpha \Theta \alpha)$	$\alpha \wedge \beta - (\alpha \Theta \beta)$	$-(\alpha \wedge \beta) \Theta \alpha$
β	$-\alpha \wedge \beta - (\alpha \Theta \beta)$	$-(\beta \Theta \beta)$	$-(\alpha \wedge \beta) \Theta \beta$
$\alpha \wedge \beta$	$(\alpha \wedge \beta) \Theta \alpha$	$(\alpha \wedge \beta) \Theta \beta$	$-(\alpha \wedge \beta) \Theta (\alpha \wedge \beta)$

11.21

The norm of a quaternion

Let \mathbf{Q} be a quaternion given in basis-free form as an element of a 2-space under the hypercomplex product.

$$\mathbf{Q} = \mathbf{a} + \alpha + \alpha \wedge \beta \quad 11.22$$

Here, \mathbf{a} is a scalar, α is a 1-element and $\alpha \wedge \beta$ is congruent to any 2-element of the space.

The norm of \mathbf{Q} is denoted $n[\mathbf{Q}]$ and given as the hypercomplex product of \mathbf{Q} with its conjugate \mathbf{Q}^c . expanding using formula 11.5 gives:

$$\begin{aligned} n[\mathbf{Q}] &= (\mathbf{a} + \alpha + \alpha \wedge \beta) \circ (\mathbf{a} - \alpha - \alpha \wedge \beta) = \\ &= \mathbf{a}^2 - (\alpha + \alpha \wedge \beta) \circ (\alpha + \alpha \wedge \beta) \end{aligned}$$

Expanding the last term gives:

$$n[\mathbf{Q}] = \mathbf{a}^2 - \alpha \circ \alpha - \alpha \circ (\alpha \wedge \beta) - (\alpha \wedge \beta) \circ \alpha - (\alpha \wedge \beta) \circ (\alpha \wedge \beta)$$

From table 11.21 we see that:

$$\alpha \circ (\alpha \wedge \beta) = -(\alpha \wedge \beta) \Theta \alpha \quad (\alpha \wedge \beta) \circ \alpha = (\alpha \wedge \beta) \Theta \alpha$$

Whence:

$$\alpha \circ (\alpha \wedge \beta) = -(\alpha \wedge \beta) \Theta \alpha$$

Using table 11.21 again then allows us to write the norm of a quaternion either in terms of the hypercomplex product or the inner product.

$$n[\mathcal{Q}] = a^2 - \alpha \circ \alpha - (\alpha \wedge \beta) \circ (\alpha \wedge \beta)$$

11.23

$$n[\mathcal{Q}] = a^2 + \alpha \Theta \alpha + (\alpha \wedge \beta) \Theta (\alpha \wedge \beta)$$

11.24

In terms of \mathbf{i} , \mathbf{j} , and \mathbf{k}

$$\mathcal{Q} = a + b \mathbf{i} + c \mathbf{j} + d \mathbf{k}$$

$$\begin{aligned} n[\mathcal{Q}] &= a^2 - (b \mathbf{i} + c \mathbf{j}) \circ (b \mathbf{i} + c \mathbf{j}) - (d \mathbf{k}) \circ (d \mathbf{k}) \\ &= a^2 - b^2 (\mathbf{i} \circ \mathbf{i}) - c^2 (\mathbf{j} \circ \mathbf{j}) - d^2 (\mathbf{k} \circ \mathbf{k}) \\ &= a^2 + b^2 (\mathbf{i} \Theta \mathbf{i}) + c^2 (\mathbf{j} \Theta \mathbf{j}) + d^2 (\mathbf{k} \Theta \mathbf{k}) \\ &= a^2 + b^2 + c^2 + d^2 \end{aligned}$$

The Cayley-Dickson algebra

If we set α and β to be orthogonal in the general hypercomplex product table 11.21, we retrieve the multiplication table for the Cayley-Dickson algebra with 4 generators and 2 parameters.

1	α	β	$\alpha \wedge \beta$
α	$-(\alpha \Theta \alpha)$	$\alpha \wedge \beta$	$-(\alpha \Theta \alpha) \beta$
β	$-\alpha \wedge \beta$	$-(\beta \Theta \beta)$	$(\beta \Theta \beta) \alpha$
$\alpha \wedge \beta$	$(\alpha \Theta \alpha) \beta$	$-(\beta \Theta \beta) \alpha$	$-(\alpha \Theta \alpha) (\beta \Theta \beta)$

11.25

In the notation we have used, the generators are $\mathbf{1}$, α , β , and $\alpha \wedge \beta$. The parameters are $\alpha \Theta \alpha$ and $\beta \Theta \beta$.

11.6 The Norm of Grassmann number

The norm

The first property that \mathbb{R} , \mathbb{C} , \mathbb{Q} , and \mathbb{O} possess in common is that of being normed.

The *norm* of a Grassmann number \mathbf{X} is denoted $n[\mathbf{X}]$ and has been defined in 11.5 as the hypercomplex product of \mathbf{X} with its conjugate.

$$n[\mathbf{X}] = \mathbf{X} \circ \mathbf{X}^c = (\mathbf{X}_b + \mathbf{X}_s) \circ (\mathbf{X}_b - \mathbf{X}_s) = \mathbf{X}_b^2 - \mathbf{X}_s \circ \mathbf{X}_s$$

The norm of an m -element α is then denoted $n[\alpha]$ and defined as the hypercomplex product of α with its conjugate α^c . If α is not scalar, then α^c is simply $-\alpha$. Hence

$$n[\alpha] = \alpha \circ \alpha^c = -\alpha \circ \alpha \quad m \neq 0 \quad 11.26$$

We require that the norm of any element be a *positive scalar* quantity.

The norm of a simple m -element

If α is *simple*, then the generalized Grassmann product $\alpha \Delta_\lambda \alpha$ is zero for all λ except λ equal to m , in which case the generalized product (and hence the hypercomplex product) becomes an inner product. Thus for m not zero we have:

$$\alpha \circ \alpha = \sigma^{m,m,m} \alpha \Theta \alpha \quad m \neq 0$$

Equation 11.26 then implies that for the norm to be a positive scalar quantity we must have $\sigma^{m,m,m} = -1$.

$$n[\alpha] = \alpha \Theta \alpha \quad \alpha \text{ simple} \quad 11.27$$

$$\sigma^{m,m,m} = -1 \quad 11.28$$

The norm of a scalar \mathbf{a} is just $\mathbf{a}^2 = \mathbf{a} \Theta \mathbf{a}$, so this formula 11.27 applies also for m equal to zero.

The skew-symmetry of products of elements of different grade

We consider a general Grassmann number $\mathbf{X} = \mathbf{a} + \mathbf{A}$, where \mathbf{a} is its body (scalar component) and \mathbf{A} is its soul (non-scalar component). We define the norm $n[\mathbf{X}]$ of \mathbf{X} in the standard way as the hypercomplex product of \mathbf{X} with its conjugate \mathbf{X}^c .

$$n[\mathbf{X}] = \mathbf{X} \circ \mathbf{X}_c = (\mathbf{a} + \mathbf{A}) \circ (\mathbf{a} - \mathbf{A}) = \mathbf{a}^2 - \mathbf{A} \circ \mathbf{A}$$

The norm of \mathbf{X} can be written in terms of \mathbf{X} as:

$$n[\mathbf{X}] = \text{Body}[\mathbf{X}]^2 - \text{Soul}[\mathbf{X}] \circ \text{Soul}[\mathbf{X}]$$

To investigate the norm further we look at the product $\mathbf{A} \circ \mathbf{A}$. Suppose \mathbf{A} is the sum of two (not necessarily simple) elements of different grade: an m -element α and a k -element β . Then $\mathbf{A} \circ \mathbf{A}$ becomes:

$$\mathbf{A} \circ \mathbf{A} = \left(\alpha + \beta \right) \circ \left(\alpha + \beta \right) = \alpha \circ \alpha + \alpha \circ \beta + \beta \circ \alpha + \beta \circ \beta$$

We would *like* the norm of a general Grassmann number to be a scalar quantity. But none of the generalized product components of $\alpha \circ \beta$ or $\beta \circ \alpha$ can be scalar if m and k are different, so we choose to make $\beta \circ \alpha$ equal to $-\alpha \circ \beta$ as a fundamental defining axiom of the hypercomplex product, thus eliminating both products from the expression for the norm. This requirement of skew-symmetry can always be satisfied because, since the grades are different, the two products are always distinguishable.

$$\alpha \circ \beta = -\beta \circ \alpha$$

11.29

The norm of a Grassmann number in terms of hypercomplex products

More generally, suppose \mathbf{A} were the sum of several elements of different grade.

$$\mathbf{A} = \alpha + \beta + \gamma + \dots$$

The skew-symmetry axiom then allows us to write $\mathbf{A} \circ \mathbf{A}$ as the sum of hypercomplex squares of the components of different grade of \mathbf{A} .

$$\mathbf{A} \circ \mathbf{A} = \left(\alpha + \beta + \gamma + \dots \right) \circ \left(\alpha + \beta + \gamma + \dots \right) = \alpha \circ \alpha + \beta \circ \beta + \gamma \circ \gamma + \dots$$

And hence the norm is expressed simply as

$$n[\mathbf{X}] = \mathbf{a}^2 - \alpha \circ \alpha - \beta \circ \beta - \gamma \circ \gamma - \dots$$

$$n[\mathbf{a} + \alpha + \beta + \gamma + \dots] = \mathbf{a}^2 - \alpha \circ \alpha - \beta \circ \beta - \gamma \circ \gamma - \dots$$

11.30

The norm of a Grassmann number of simple components

Consider a Grassmann number \mathbf{A} in which the elements α, β, \dots of each grade are simple.

$$\mathbf{A} = \alpha + \beta + \gamma + \dots$$

Then by 11.28 and 11.30 we have that the norm of \mathbf{A} is positive and may be written as the sum of interior products of the individual simple elements.

$$n \left[\mathbf{a} + \underset{m}{\alpha} + \underset{k}{\beta} + \underset{p}{\gamma} + \dots \right] = \mathbf{a}^2 + \underset{m}{\alpha} \ominus \underset{m}{\alpha} + \underset{k}{\beta} \ominus \underset{k}{\beta} + \underset{p}{\gamma} \ominus \underset{p}{\gamma} + \dots \quad 11.31$$

The norm of a non-simple element

We now turn our attention to the case in which one of the component elements of \mathbf{x} may not be simple. This is the general case for Grassmann numbers in spaces of dimension greater than 3, and indeed is the reason why the suite of normed hypercomplex numbers ends with the Octonions. An element of any grade in a space of any dimension 3 or less must be simple. So in spaces of dimension 0, 1, 2, and 3 the norm of an element will always be a scalar.

We now focus our attention on the product $\underset{m}{\alpha} \circ \underset{m}{\alpha}$ (where $\underset{m}{\alpha}$ may or may not be simple). The following analysis will be critical to our understanding of how to generalize the notion of hypercomplex numbers to spaces of dimension greater than 3.

Suppose $\underset{m}{\alpha}$ is the sum of two simple elements $\underset{m}{\alpha}_1$ and $\underset{m}{\alpha}_2$. The product $\underset{m}{\alpha} \circ \underset{m}{\alpha}$ then becomes:

$$\underset{m}{\alpha} \circ \underset{m}{\alpha} = \left(\underset{m}{\alpha}_1 + \underset{m}{\alpha}_2 \right) \circ \left(\underset{m}{\alpha}_1 + \underset{m}{\alpha}_2 \right) = \underset{m}{\alpha}_1 \circ \underset{m}{\alpha}_1 + \underset{m}{\alpha}_1 \circ \underset{m}{\alpha}_2 + \underset{m}{\alpha}_2 \circ \underset{m}{\alpha}_1 + \underset{m}{\alpha}_2 \circ \underset{m}{\alpha}_2$$

Since $\underset{m}{\alpha}_1$ and $\underset{m}{\alpha}_2$ are simple, $\underset{m}{\alpha}_1 \circ \underset{m}{\alpha}_1$ and $\underset{m}{\alpha}_2 \circ \underset{m}{\alpha}_2$ are scalar, and can be written:

$$\underset{m}{\alpha}_1 \circ \underset{m}{\alpha}_1 + \underset{m}{\alpha}_2 \circ \underset{m}{\alpha}_2 = - \underset{m}{\alpha}_1 \ominus \underset{m}{\alpha}_1 - \underset{m}{\alpha}_2 \ominus \underset{m}{\alpha}_2$$

The expression of the remaining terms $\underset{m}{\alpha}_1 \circ \underset{m}{\alpha}_2$ and $\underset{m}{\alpha}_2 \circ \underset{m}{\alpha}_1$ in terms of exterior and interior products is more interesting and complex. We discuss it in the next section.

11.7 Products of two different elements of the same grade

The symmetrized sum of two m -elements

We now investigate the remaining terms, products of *two different elements of the same grade*. We have left this discussion to last as these types of term introduce the most complexity into the nature of the hypercomplex product. It is their potential to introduce non-scalar terms into the (usually defined) norms of hypercomplex numbers of order higher than the Octonions that makes them of most interest.

Returning to our definition of the hypercomplex product and specializing it to the case where both elements are of the same grade we get:

$$\alpha_1 \circ_m \alpha_2 = \sum_{\lambda=0}^m \sigma^{m,\lambda,m} \alpha_1 \Delta_{\lambda} \alpha_2$$

Reversing the order of the factors in the hypercomplex product gives:

$$\alpha_2 \circ_m \alpha_1 = \sum_{\lambda=0}^m \sigma^{m,\lambda,m} \alpha_2 \Delta_{\lambda} \alpha_1$$

The generalized products on the right may now be reversed together with the change of sign $(-1)^{(m-\lambda)} = (-1)^{(m-\lambda)}$.

$$\alpha_2 \circ_m \alpha_1 = \sum_{\lambda=0}^m (-1)^{(m-\lambda)} \sigma^{m,\lambda,m} \alpha_1 \Delta_{\lambda} \alpha_2$$

The sum $\alpha_1 \circ_m \alpha_2 + \alpha_2 \circ_m \alpha_1$ may then be written finally as:

$$\alpha_1 \circ_m \alpha_2 + \alpha_2 \circ_m \alpha_1 = \sum_{\lambda=0}^m (1 + (-1)^{(m-\lambda)}) \sigma^{m,\lambda,m} \alpha_1 \Delta_{\lambda} \alpha_2 \quad 11.32$$

Because we will need to refer to this sum of products subsequently, we will call it the *symmetrized sum* of two m -elements (because the sum of products does not change if the order of the elements is reversed).

Symmetrized sums for elements of different grades

For two (in general, different) simple elements of the same grade α_1 and α_2 we have shown above that:

$$\alpha_1 \circ_m \alpha_2 + \alpha_2 \circ_m \alpha_1 = \sum_{\lambda=0}^m (1 + (-1)^{m-\lambda}) \sigma^{m,\lambda,m} \alpha_1 \Delta_{\lambda} \alpha_2$$

The term $1 + (-1)^{m-\lambda}$ will be 2 for m and λ both even or both odd, and zero otherwise. To get a clearer picture of what this formula entails, we write it out explicitly for the lower values of m . Note that we use the fact already established that $\sigma^{m,m,m} = -1$.

◆ $m = 0$

$$a \circ b + b \circ a = 2 a b \quad 11.33$$

◆ $m = 1$

$$\alpha_1 \circ \alpha_2 + \alpha_2 \circ \alpha_1 == -2 \alpha_1 \Theta \alpha_2 \quad 11.34$$

◆ $m = 2$

$$\alpha_1 \circ \alpha_2 + \alpha_2 \circ \alpha_1 == 2 \left(\overset{2,0,2}{\sigma} \alpha_1 \wedge \alpha_2 - \alpha_1 \Theta \alpha_2 \right) \quad 11.35$$

◆ $m = 3$

$$\alpha_1 \circ \alpha_2 + \alpha_2 \circ \alpha_1 == 2 \left(\overset{3,1,3}{\sigma} \alpha_1 \Delta \alpha_2 - \alpha_1 \Theta \alpha_2 \right) \quad 11.36$$

◆ $m = 4$

$$\alpha_1 \circ \alpha_2 + \alpha_2 \circ \alpha_1 == 2 \left(\overset{4,0,4}{\sigma} \alpha_1 \wedge \alpha_2 + \overset{4,2,4}{\sigma} \alpha_1 \Delta \alpha_2 - \alpha_1 \Theta \alpha_2 \right) \quad 11.37$$

The body of a symmetrized sum

We now break this sum up into its body (scalar component) and soul (non-scalar components).

The body of $\alpha_1 \circ \alpha_2 + \alpha_2 \circ \alpha_1$ is the inner product term in which λ becomes equal to m , that is: $2 \overset{m,m,m}{\sigma} \left(\alpha_1 \Theta \alpha_2 \right)$.

And since constraint 11.28 requires that $\overset{m,m,m}{\sigma} == -1$, the body of $\alpha \circ \alpha$ becomes:

$$\begin{aligned} \mathbf{Body} \left[\alpha \circ \alpha \right] &== \alpha_1 \circ \alpha_1 + \mathbf{Body} \left[\alpha_1 \circ \alpha_2 + \alpha_2 \circ \alpha_1 \right] + \alpha_2 \circ \alpha_2 \\ &== - \left(\alpha_1 \Theta \alpha_1 + 2 \alpha_1 \Theta \alpha_2 + \alpha_2 \Theta \alpha_2 \right) == - \alpha \Theta \alpha \end{aligned}$$

Hence, even if a component element like α is not simple, the body of its hypercomplex square is given by the same expression as if it were simple, that is, its inner product square.

$$\text{Body} \left[\alpha \circ \alpha \right]_{\mathbf{m}} = - \alpha \Theta \alpha_{\mathbf{m}}$$

11.38

The soul of a symmetrized sum

The soul of $\alpha \circ \alpha_{\mathbf{m}}$ is then the non-scalar residue given by the formula:

$$\text{Soul} \left[\alpha \circ \alpha \right]_{\mathbf{m}} = \sum_{\lambda=0}^{m-2} (1 + (-1)^{(m-\lambda)}) \sigma_{\mathbf{m}, \lambda, \mathbf{m}}^{\alpha_1 \Delta \alpha_2}$$

11.39

(Here we have used the fact that because of the coefficient $(1 + (-1)^{(m-\lambda)})$, the terms with λ equal to $m-1$ is always zero, and we only sum to $m-2$).

For convenience in generating examples of this expression, we temporarily define the soul of $\alpha \circ \alpha_{\mathbf{m}}$ as a function of m , and denote it $\eta[\mathbf{m}]$. To make it easier to read we convert the generalized products where possible to exterior and interior products.

$$\eta[\mathbf{m}_] := \sum_{\lambda=0}^{m-2} (1 + (-1)^{(m-\lambda)}) \sigma_{\mathbf{m}, \lambda, \mathbf{m}}^{\alpha_1 \Delta \alpha_2} //$$

GeneralizedProductSimplify

With this function we can make a palette of souls for various values of m . We do this below for m equals 1 to 6 to cover spaces of dimension from 1 to 6.

PaletteTemplate[Table[{m, η[m]}, {m, 1, 6}]]

1	0
2	$2 \sigma_{2, 0, 2}^{\alpha_1 \wedge \alpha_2}$
3	$2 \left(\alpha_1 \Delta \alpha_2 \right)_3^{\sigma, 1, 3}$
4	$2 \left(\alpha_1 \Delta \alpha_2 \right)_4^{\sigma, 2, 4} + 2 \sigma_{4, 0, 4}^{\alpha_1 \wedge \alpha_2}$
5	$2 \left(\alpha_1 \Delta \alpha_2 \right)_5^{\sigma, 1, 5} + 2 \left(\alpha_1 \Delta \alpha_2 \right)_5^{\sigma, 3, 5}$
6	$2 \left(\alpha_1 \Delta \alpha_2 \right)_6^{\sigma, 2, 6} + 2 \left(\alpha_1 \Delta \alpha_2 \right)_6^{\sigma, 4, 6} + 2 \sigma_{6, 0, 6}^{\alpha_1 \wedge \alpha_2}$

Of particular note is the soul of the symmetrized sum of two different simple 2-elements.

$$2^{2,0,2} \sigma \alpha_1 \wedge \alpha_2$$

This 4-element is the critical potentially non-scalar residue in a norm calculation. It is always zero in spaces of dimension 2, or 3. *Hence the norm of a Grassmann number in these spaces under the hypercomplex product is guaranteed to be scalar.*

In a space of 4 dimensions, on the other hand, this element may not be zero. *Hence the space of dimension 3 is the highest-dimensional space in which the norm of a Grassmann number is guaranteed to be scalar.*

We have already seen that Grassmann numbers in a 2-space under the hypercomplex product generate the quaternions. In the next section we shall see that Grassmann numbers in a 3-space under the hypercomplex product generate the Octonions. The Octonions therefore will be the last hypercomplex system with a scalar norm.

Summary of results of this section

Consider a general Grassmann number \mathbf{X} .

$$\mathbf{X} = \mathbf{a} + \alpha_m + \beta_k + \gamma_p + \dots$$

Here, we suppose \mathbf{a} to be scalar, and the rest of the terms to be nonscalar simple or non-simple elements.

The generalized hypercomplex norm $n[\mathbf{X}]$ of \mathbf{X} may be written as:

$$n[\mathbf{X}] = \mathbf{a}^2 - \alpha_m \circ \alpha_m - \beta_k \circ \beta_k - \gamma_p \circ \gamma_p - \dots$$

The hypercomplex square $\alpha_m \circ \alpha_m$ of an element, has in general, both a body and a soul.

The body of $\alpha_m \circ \alpha_m$ is the negative of the inner product.

$$\text{Body}[\alpha_m \circ \alpha_m] = -\alpha_m \Theta_m \alpha_m$$

The soul of $\alpha_m \circ \alpha_m$ depends on the terms of which it is composed. If

$$\alpha_m = \alpha_1 + \alpha_2 + \alpha_3 + \dots \text{ then}$$

$$\text{Soul}[\alpha_m \circ \alpha_m] = \sum_{\lambda=0}^{m-2} (1 + (-1)^{(m-\lambda)}) \sigma^{m,\lambda,m} \left(\alpha_1 \Delta_{\lambda} \alpha_2 + \alpha_1 \Delta_{\lambda} \alpha_3 + \alpha_2 \Delta_{\lambda} \alpha_3 + \dots \right)$$

If the component elements of different grade of \mathbf{X} are simple, then its soul is zero, and its norm becomes the scalar:

$$n[\mathbf{X}] = a^2 + \alpha_m \Theta \alpha_m + \beta_k \Theta \beta_k + \gamma_p \Theta \gamma_p + \dots$$

It is only in 3-space that we are guaranteed that all the components of a Grassmann number are simple. *Therefore it is only in 3-space that we are guaranteed that the norm of a Grassmann number under the hypercomplex product is scalar.*

11.8 Octonions

To be completed