8 Exploring Mechanics

8.1 Introduction

Grassmann algebra applied to the field of mechanics performs an interesting synthesis between what now seem to be regarded as disparate concepts. In particular we will explore the synthesis it can effect between the concepts of force and moment; velocity and angular velocity; and linear momentum and angular momentum.

This synthesis has an important concomitant, namely that the form in which the mechanical entities are represented, is for many results of interest, *independent of the dimension of the space involved*. It may be argued therefore, that such a representation is more fundamental than one specifically requiring a three-dimensional context (as indeed does that which uses the Gibbs-Heaviside vector algebra).

This is a more concrete result than may be apparent at first sight since the form, as well as being valid for spaces of dimension three or greater, is also valid for spaces of dimension zero, one or two. Of most interest, however, is the fact that the complementary form of a mechanical entity takes on a different form depending on the number of dimensions concerned. For example, the velocity of a rigid body is the sum of a bound vector and a bivector in a space of any dimensions. Its complement is dependent on the dimension of the space, but in each case it may be viewed as representing the points of the body (if they exist) which have *zero velocity*. In three dimensions the complementary velocity can specify the *axis of rotation* of the rigid body, while in two dimensions it can specify the *centre* (*point*) of rotation.

Furthermore, some results in mechanics (for example, Newton's second law or the conditions of equilibrium of a rigid body) will be shown *not to require the space to be a metric space*. On the other hand, use of the Gibbs-Heaviside 'cross product' to express angular momentum or the moment condition immediately supposes the space to possess a metric.

Mechanics as it is known today is in the strange situation of being a field of mathematical physics in which *location* is very important, and of which the calculus traditionally used (being vectorial) can take no proper account. As already discussed in Chapter 1, one may take the example of the concept of *force*. A (physical) force is not satisfactorily represented by a vector, yet contemporary practice is still to use a vector calculus for this task. To patch up this inadequacy the concept of moment is introduced and the conditions of equilibrium of a rigid body augmented by a condition on the sum of the moments. The justification for this condition is often not well treated in contemporary texts.

Many of these texts will of course rightly state that forces are not (represented by) free vectors and yet will proceed to use the Gibbs-Heaviside calculus to denote and manipulate them. Although confusing, this inaptitude is usually offset by various comments attached to the symbolic descriptions and calculations. For example, a position is represented by a 'position vector'. A position vector is described as a (free?) vector with its tail (fixed?) to the origin (point?) of the coordinate system. The coordinate system itself is supposed to consist of an origin *point* and a number of basis vectors. But whereas the vector calculus can cope with the vectors, it cannot cope with the origin. This confusion between vectors, free vectors, bound vectors, sliding vectors and position vectors would not occur if the calculus used to describe them were a calculus of *position* (points) as well as *direction* (vectors).

In order to describe the phenomena of mechanics in purely vectorial terms it has been necessary therefore to devise the notions of couple and moment as notions almost distinct from that of force, thus effectively splitting in two all the results of mechanics. In traditional mechanics, to every 'linear' quantity: force, linear momentum, translational velocity, etc., corresponds an 'angular' quantity: moment, angular momentum, angular velocity.

In this chapter we will show that by representing mechanical quantities correctly in terms of elements of a Grassmann algebra this dichotomy disappears and mechanics takes on a more unified form. In particular we will show that there exists a screw $\mathbb{F}$ representing (*including* moments) a system of forces (remember that a system of forces in three dimensions is not necessarily replaceable by a single force); a screw $\mathbb{L}$ representing the momentum of a system of bodies (linear *and* angular), and a screw $\mathbb{V}$ representing the velocity of a rigid body (linear and angular): all invariant combinations of the linear and angular components. For example, the velocity $\mathbb{V}$ is a complete characterization of the kinematics of the rigid body independent of any particular point on the body used to specify its motion. Expressions for work, power and kinetic energy of a system of forces and a rigid body will be shown to be determinable by an interior product between the relevant screws. For example, the interior product of $\mathbb{F}$ with $\mathbb{V}$ will give the power of the system of forces acting on the rigid body, that is, the *sum* of the translational and rotational powers.

Historical Note

The application of the *Ausdehnungslehre* to mechanics has obtained far fewer proponents than might be expected. This may be due to the fact that Grassmann himself was late going into the subject. His '*Die Mechanik nach den principien der Ausdehnungslehre*' (1877) was written just a few weeks before his death. Furthermore, in the period before his ideas became completely submerged by the popularity of the Gibbs-Heaviside system (around the turn of the century) there were few people with sufficient understanding of the *Ausdehnungslehre* to break new ground using its methods.

There are only three people who have written substantially in English using the original concepts of the *Ausdehnungslehre*: Edward Wyllys Hyde (1888), Alfred North Whitehead (1898), and Henry James Forder (1941). Each of them has discussed the theory of screws in more or less detail, but none has addressed the complete field of mechanics. The principal works in other languages are in German, and apart from Grassmann's paper in 1877 mentioned above, are the book by Jahnke (1905) which includes applications to mechanics, and the short monograph by Lotze (1922) which lays particular emphasis on rigid body mechanics.

8.2 Force

Representing force

The notion of force as used in mechanics involves the concepts of magnitude, sense, and line of action. It will be readily seen in what follows that such a physical entity may be faithfully represented by a *bound vector*. Similarly, all the usual properties of systems of forces are faithfully represented by the analogous properties of sums of bound vectors. For the moment it is not important whether the space has a metric, or what its dimension is.

Let $\mathbb{F}$ denote a force. Then:

$$\mathbb{F} == \mathbf{P} \wedge \mathbf{f} \tag{8.1}$$

The vector **f** is called the *force vector* and expresses the sense and direction of the force. In a metric space it would also express its magnitude.

The point **P** is any point on the line of action of the force. It can be expressed as the sum of the origin point $\mathbb{O}$ and the position vector **v** of the point.

This simple representation delineates clearly the role of the (free) vector **f**. The vector **f** represents all the properties of the force *except* the position of its line of action. The operation $\mathbf{P}\wedge$ has the effect of 'binding' the force vector to the line through **P**.

On the other hand the expression of a bound vector as a product of points is also useful when expressing certain types of forces, for example gravitational forces. Newton's law of gravitation for the force exerted on a *point* mass $\mathbb{m}_1$ (weighted point) by a *point* mass $\mathbb{m}_2$ may be written:

$$\mathbb{F}_{12} == \frac{G\, \mathbb{m}_1 \wedge \mathbb{m}_2}{R^2} \tag{8.2}$$

Note that this expression correctly changes sign if the masses are interchanged.

Systems of forces

A *system of forces* may be represented by a *sum of bound vectors*.

$$\mathbb{F} == \sum \mathbb{F}_i == \sum \mathbf{P}_i \wedge \mathbf{f}_i \tag{8.3}$$

A sum of bound vectors is not necessarily reducible to a bound vector: a system of forces is not necessarily reducible to a single force. However, a sum of bound vectors is in general reducible to the sum of a bound vector and a bivector. This is done simply by adding and subtracting the term $\mathbf{P} \wedge (\sum \mathbf{f}_i)$ to and from the sum [8.3].

$$\mathbb{F} == \mathbf{P} \wedge \left(\sum \mathbf{f_i}\right) + \sum (\mathbf{P_i} - \mathbf{P}) \wedge \mathbf{f_i} \tag{8.4}$$

This 'adding and subtracting' operation is a common one in our treatment of mechanics. We call it *referring the sum to the point* $\mathbf{P}$.

Note that although the expression for $\mathbb{F}$ now involves the point $\mathbf{P}$, it is completely independent of $\mathbf{P}$ since the terms involving $\mathbf{P}$ cancel. The sum may be said to be *invariant* with respect to any point used to express it.

Examining the terms in the expression 8.4 above we can see that since $\sum \mathbf{f_i}$ is a vector ($\mathbf{f}$, say) the first term reduces to the bound vector $\mathbf{P} \wedge \mathbf{f}$ representing a force through the chosen point $\mathbf{P}$. The vector $\mathbf{f}$ is called the *resultant force vector*.

$$\mathbf{f} == \sum \mathbf{f_i}$$

The second term is a sum of bivectors of the form $(\mathbf{P_i} - \mathbf{P}) \wedge \mathbf{f_i}$. Such a bivector may be seen to faithfully represent a *moment*: specifically, *the moment of the force represented by* $\mathbf{P_i} \wedge \mathbf{f_i}$ *about the point* $\mathbf{P}$. Let this moment be denoted $\mathbb{G}_{\mathbf{iP}}$.

$$\mathbb{G}_{\mathbf{iP}} == (\mathbf{P_i} - \mathbf{P}) \wedge \mathbf{f_i}$$

To see that it is more reasonable that a moment be represented as a bivector rather than as a vector, one has only to consider that the physical dimensions of a moment are the *product* of a length and a force unit.

The expression for moment above clarifies distinctly how it can arise from two located entities: the bound vector $\mathbf{P_i} \wedge \mathbf{f_i}$ and the point $\mathbf{P}$, and yet itself be a 'free' entity. The bivector, it will be remembered, has no concept of location associated with it. This has certainly been a source of confusion among students of mechanics using the usual Gibbs-Heaviside three-dimensional vector calculus. It is well known that the moment of a force about a point does not possess the property of location, *and yet it still depends on that point*. While notions of 'free' and 'bound' are not properly mathematically characterized, this type confusion is likely to persist.

The second term in [8.4] representing the sum of the moments of the forces about the point $\mathbf{P}$ will be denoted:

$$\mathbb{G}_{\mathbf{P}} == \sum (\mathbf{P_i} - \mathbf{P}) \wedge \mathbf{f_i}$$

Then, *any system of forces may be represented by the sum of a bound vector and a bivector*.

$$\mathbb{F} == \mathbf{P} \wedge \mathbf{f} + \mathbb{G}_{\mathbf{P}} \tag{8.5}$$

The bound vector $\mathbf{P} \wedge \mathbf{f}$ represents a force through an arbitrary point $\mathbf{P}$ with force vector equal to the sum of the force vectors of the system.

The bivector $\mathbb{G}_{\mathbf{P}}$ represent the sum of the moments of the forces about the same point $\mathbf{P}$.

Suppose the system of forces be referred to some other point $\mathbf{P}^*$. The system of forces may then be written in either of the two forms:

$$\mathbf{P} \wedge \mathbf{f} + \mathbb{G}_{\mathbf{P}} == \mathbf{P}^* \wedge \mathbf{f} + \mathbb{G}_{\mathbf{P}^*}$$

Solving for $\mathbb{G}_{\mathbf{P}^*}$ gives us a formula relating the moment sum about different points.

$$\mathbb{G}_{\mathbf{P}^*} == \mathbb{G}_{\mathbf{P}} + (\mathbf{P} - \mathbf{P}^*) \wedge \mathbf{f}$$

If the bound vector term $\mathbf{P} \wedge \mathbf{f}$ in formula 8.5 is zero then the system of forces is called a *couple*. If the bivector term $\mathbb{G}_{\mathbf{P}}$ is zero then the system of forces *reduces to a single force*.

Equilibrium

If a sum of forces is zero, that is:

$$\mathbb{F} == \mathbf{P} \wedge \mathbf{f} + \mathbb{G}_{\mathbf{P}} == 0$$

then it is straightforward to show that each of $\mathbf{P} \wedge \mathbf{f}$ and $\mathbb{G}_{\mathbf{P}}$ must be zero. Indeed if such were not the case then the bound vector would be equal to the negative of the bivector, a possibility excluded by the implicit presence of the origin in a bound vector, and its absence in a bivector. Furthermore $\mathbf{P} \wedge \mathbf{f}$ being zero implies $\mathbf{f}$ is zero.

These considerations lead us directly to the basic theorem of statics. *For a body to be in equilibrium, the sum of the forces must be zero*.

$$\boxed{\mathbb{F} == \sum \mathbb{F}_{\mathbf{i}} == 0} \tag{8.6}$$

This one expression encapsulates *both* the usual conditions for equilibrium of a body, that is: that the sum of the force vectors be zero; and that the sum of the moments of the forces about an arbitrary point $\mathbf{P}$ be zero.

Force in a metric 3-plane

In a 3-plane the bivector $\mathbb{G}_{\mathbf{P}}$ is necessarily simple. In a metric 3-plane, the vector-space complement of the simple bivector $\mathbb{G}_{\mathbf{P}}$ is just the usual moment *vector* of the three-dimensional Gibbs-Heaviside vector calculus.

In three dimensions, the complement of an exterior product is equivalent to the usual cross product.

$$\overrightarrow{\mathbb{G}_{\mathbf{P}}} == \sum (\mathbf{P}_{\mathbf{i}} - \mathbf{P}) \times \mathbf{f}_{\mathbf{i}}$$

Thus, a system of forces in a metric 3-plane may be reduced to a single force through an arbitrarily chosen point $\mathbf{P}$ *plus* the *vector-space complement* of the usual moment vector about $\mathbf{P}$.

$$\boxed{\mathbb{F} == \mathbf{P} \wedge \mathbf{f} + \overrightarrow{\sum (\mathbf{P}_{\mathbf{i}} - \mathbf{P}) \times \mathbf{f}_{\mathbf{i}}}} \tag{8.7}$$

8.3 Momentum

The velocity of a particle

Suppose a point P with position vector ν, that is $P == \mathbb{O} + \nu$.

Since the origin is fixed, the velocity $\overset{\circ}{P}$ of the point P is clearly a vector given by the time-derivative of the position vector of P.

$$\boxed{\overset{\circ}{P} == \frac{d\,P}{d\,t} == \frac{d\,\nu}{d\,t} == \overset{\circ}{\nu}} \qquad 8.8$$

Representing momentum

As may be suspected from Newton's second law, momentum is of the same tensorial nature as force. The momentum of a particle is represented by a bound vector as is a force. The momentum of a system of particles (or of a rigid body, or of a system of rigid bodies) is representable by a sum of bound vectors as is a system of forces.

The momentum of a particle comprises three factors: the *mass*, the *position*, and the *velocity* of the particle. The mass is represented by a scalar, the position by a point, and the velocity by a vector.

$$\boxed{\mathbb{L} == P \wedge \left(m\,\overset{\circ}{P}\right)} \qquad 8.9$$

Here	$\mathbb{L}$	is the particle momentum.
	m	is the particle mass.
	P	is the particle position point.
	$\dot{P}$	is the particle velocity.
	$m\,P$	is the particle point mass.
	$m\,\overset{\circ}{P}$	is the particle momentum vector.

The momentum of a particle may be viewed either as a momentum vector bound to a line through the position of the particle, or as a velocity vector bound to a line through the point mass. The simple description [8.9] delineates clearly the role of the (free) vector $m\,\overset{\circ}{P}$. It represents all the properties of the momentum of a particle *except* its position.

The momentum of a system of particles

The momentum of a system of particles is denoted by $\mathbb{L}$, and may be represented by the sum of bound vectors:

$$\boxed{\mathbb{L} == \sum \mathbb{L}_i == \sum \mathbf{P_i} \wedge \left(\mathbf{m_i}\, \overset{\circ}{\mathbf{P}}_\mathbf{i}\right)} \tag{8.10}$$

A sum of bound vectors is not necessarily reducible to a bound vector: the momentum of a system of particles is not necessarily reducible to the momentum of a single particle. However, a sum of bound vectors is in general reducible to the sum of a bound vector and a bivector. This is done simply by adding and subtracting the terms $\mathbf{P} \wedge \sum \mathbf{m_i}\, \overset{\circ}{\mathbf{P}}_\mathbf{i}$ to and from the sum [8.10].

$$\boxed{\mathbb{L} == \mathbf{P} \wedge \sum \mathbf{m_i}\, \overset{\circ}{\mathbf{P}}_\mathbf{i} + \sum (\mathbf{P_i} - \mathbf{P}) \wedge \left(\mathbf{m_i}\, \overset{\circ}{\mathbf{P}}_\mathbf{i}\right)} \tag{8.11}$$

It cannot be too strongly emphasized that although the momentum of the system has now been referred to the point $\mathbf{P}$, it is completely independent of $\mathbf{P}$.

Examining the terms in formula 8.11 we see that since $\sum \mathbf{m_i}\, \overset{\circ}{\mathbf{P}}_\mathbf{i}$ is a vector ($\mathbf{l}$, say) the first term reduces to the bound vector $\mathbf{P} \wedge \mathbf{l}$ representing the (linear) momentum of a 'particle' situated at the point $\mathbf{P}$. The vector $\mathbf{l}$ is called the *linear momentum of the system*. The 'particle' may be viewed as a particle with mass equal to the total mass of the system and with velocity equal to the velocity of the centre of mass.

$$\mathbf{l} == \sum \mathbf{m_i}\, \overset{\circ}{\mathbf{P}}_\mathbf{i}$$

The second term is a sum of bivectors of the form $\sum (\mathbf{P_i} - \mathbf{P}) \wedge \left(\mathbf{m_i}\, \overset{\circ}{\mathbf{P}}_\mathbf{i}\right)$. Such a bivector may be seen to faithfully represent the *moment of momentum* or *angular momentum* of the system of particles about the point $\mathbf{P}$. Let this moment of momentum for particle *i* be denoted $\mathbf{H_{iP}}$.

$$\mathbf{H_{iP}} == (\mathbf{P_i} - \mathbf{P}) \wedge \left(\mathbf{m_i}\, \overset{\circ}{\mathbf{P}}_\mathbf{i}\right)$$

The expression for angular momentum $\mathbf{H_{ip}}$ above clarifies distinctly how it can arise from two *located* entities: the bound vector $\mathbf{P_i} \wedge \left(\mathbf{m_i}\, \overset{\circ}{\mathbf{P}}_\mathbf{i}\right)$ and the point $\mathbf{P}$, and yet itself be a 'free' entity. Similarly to the notion of moment, the notion of angular momentum treated using the three-dimensional Gibbs-Heaviside vector calculus has caused some confusion amongst students of mechanics: the angular momentum of a particle about a point depends on the positions of the particle and of the point and yet itself has no property of location.

The second term in formula 8.11 representing the sum of the moments of momenta about the point $\mathbf{P}$ will be denoted $\mathbb{H}_\mathbf{P}$.

$$\mathbb{H}_\mathbf{P} == \sum (\mathbf{P_i} - \mathbf{P}) \wedge \left(\mathbf{m_i}\, \overset{\circ}{\mathbf{P}}_\mathbf{i}\right)$$

Then *the momentum* $\mathbb{L}$ *of any system of particles may be represented by the sum of a bound vector and a bivector*.

$$\boxed{\mathbb{L} == \mathbf{P} \wedge \mathbf{l} + \mathbb{H}_\mathbf{P}} \tag{8.12}$$

The bound vector $\mathbf{P}\wedge\mathbf{l}$ represents the linear momentum of the system referred to an arbitrary point $\mathbf{P}$.

The bivector $\mathbb{H}_{\mathbf{P}}$ represents the angular momentum of the system about the point $\mathbf{P}$.

The momentum of a system of bodies

The momentum of a system of bodies (rigid or non-rigid) is of the same form as that for a system of particles [8.12], since to calculate momentum it is not necessary to consider any constraints between the particles.

Suppose a number of bodies with momenta $\mathbb{L}_i$, then the total momentum $\sum\mathbb{L}_i$ may be written:

$$\sum\mathbb{L}_i == \sum\mathbf{P}_i\wedge\mathbf{l}_i+\sum\mathbb{H}_{\mathbf{P}_i}$$

Referring this momentum sum to a point $\mathbf{P}$ by adding and subtracting the term $\mathbf{P}\wedge\sum\mathbf{l}_i$ we obtain:

$$\sum\mathbb{L}_i == \mathbf{P}\wedge\sum\mathbf{l}_i+\sum(\mathbf{P}_i-\mathbf{P})\wedge\mathbf{l}_i+\sum\mathbb{H}_{\mathbf{P}_i}$$

It is thus natural to represent the *linear momentum vector of the system of bodies* $\mathbf{l}$ by:

$$\mathbf{l} == \sum\mathbf{l}_i$$

and the *angular momentum of the system of bodies* $\mathbb{H}_{\mathbf{P}}$ by

$$\mathbb{H}_{\mathbf{P}} == \sum(\mathbf{P}_i-\mathbf{P})\wedge\mathbf{l}_i+\sum\mathbb{H}_{\mathbf{P}_i}$$

That is, the momentum of a system of bodies is of the same form as the momentum of a system of particles. The linear momentum vector of the system is the sum of the linear momentum vectors of the bodies. The angular momentum of the system is made up of two parts: the sum of the angular momenta of the bodies about their respective chosen points; and the sum of the moments of the linear momenta of the bodies (referred to their respective chosen points) about the point $\mathbf{P}$.

Again it may be worth emphasizing that the total momentum $\mathbb{L}$ is independent of the point $\mathbf{P}$, whilst its component terms are dependent on it. However, we can easily convert the formulae to refer to some other point say $\mathbf{P}^*$.

$$\mathbb{L} == \mathbf{P}\wedge\mathbf{l}+\mathbb{H}_{\mathbf{P}} == \mathbf{P}^*\wedge\mathbf{l}+\mathbb{H}_{\mathbf{P}^*}$$

Hence the angular momentum referred to the new point is given by:

$$\mathbb{H}_{\mathbf{P}^*} == \mathbb{H}_{\mathbf{P}}+(\mathbf{P}-\mathbf{P}^*)\wedge\mathbf{l}$$

Linear momentum and the mass centre

There is a simple relationship between the linear momentum of the system, and the momentum of a 'particle' at the mass centre. The centre of mass $\mathbf{P}_{\mathbf{G}}$ of a system may be defined by:

$$M\,P_G == \sum m_i\,P_i$$

Here M is the total mass of the system.
P_i is the position of the *i*th particle or centre of mass of the *i*th body.
P_G is the position of the centre of mass of the system.

Differentiating this equation yields:

$$M\,\overset{\circ}{P}_G == \sum m_i\,\overset{\circ}{P}_i == l$$

Formula 8.12 then may then be written:

$$\mathbb{L} == P \wedge \left(M\,\overset{\circ}{P}_G\right) + \mathbb{H}_P$$

If now we refer the momentum to the centre of mass, that is, we choose P to be P_G, we can write the momentum of the system as:

$$\boxed{\mathbb{L} == P_G \wedge \left(M\,\overset{\circ}{P}_G\right) + \mathbb{H}_{P_G}} \qquad 8.13$$

Thus the momentum $\mathbb{L}$ of a system is equivalent to the momentum of a particle of mass equal to the total mass M of the system situated at the centre of mass P_G plus the angular momentum about the centre of mass.

Momentum in a metric 3-plane

In a 3-plane the bivector $\mathbb{H}_P$ is necessarily simple. In a metric 3-plane, the vector-space complement of the simple bivector $\mathbb{H}_P$ is just the usual momentum *vector* of the three-dimensional Gibbs-Heaviside vector calculus.

In three dimensions, the complement of the exterior product of two vectors is equivalent to the usual cross product.

$$\overrightarrow{\mathbb{H}_P} == \sum (P_i - P) \times l_i$$

Thus, the momentum of a system in a metric 3-plane may be reduced to the momentum of a single particle through an arbitrarily chosen point P *plus* the *vector-space complement* of the usual moment of moment vector about P.

$$\boxed{\mathbb{L} == P \wedge l + \overrightarrow{\sum (P_i - P) \times l_i}} \qquad 8.14$$

8.4 Newton's Law

Rate of change of momentum

The momentum of a system of particles has been given by [8.10] as:

$$\mathbb{L} == \sum \mathbf{P_i} \wedge \left(\mathbf{m_i}\, \mathring{\mathbf{P}}_\mathbf{i}\right)$$

The rate of change of this momentum with respect to time is:

$$\mathring{\mathbb{L}} == \sum \mathring{\mathbf{P}}_\mathbf{i} \wedge \left(\mathbf{m_i}\, \mathring{\mathbf{P}}_\mathbf{i}\right) + \sum \mathbf{P_i} \wedge \left(\mathbf{m_i}\, \overset{\circ\circ}{\mathbf{P}}_\mathbf{i}\right) + \sum \mathbf{P_i} \wedge \left(\mathring{\mathbf{m}}_\mathbf{i}\, \mathring{\mathbf{P}}_\mathbf{i}\right)$$

In what follows, it will be supposed for simplicity that the masses are not time varying, and since the first term is zero, this expression becomes:

$$\boxed{\mathring{\mathbb{L}} == \sum \mathbf{P_i} \wedge \left(\mathbf{m_i}\, \overset{\circ\circ}{\mathbf{P}}_\mathbf{i}\right)} \qquad \mathbf{8.15}$$

Consider now the system with momentum:

$$\mathbb{L} == \mathbf{P} \wedge \mathbf{l} + \mathbb{H}_\mathbf{P}$$

The rate of change of momentum with respect to time is:

$$\boxed{\mathring{\mathbb{L}} == \mathring{\mathbf{P}} \wedge \mathbf{l} + \mathbf{P} \wedge \mathring{\mathbf{l}} + \mathring{\mathbb{H}}_\mathbf{P}} \qquad \mathbf{8.16}$$

The term $\mathring{\mathbf{P}} \wedge \mathbf{l}$, or what is equivalent, the term $\mathring{\mathbf{P}} \wedge \left(\mathbf{M}\, \mathring{\mathbf{P}}_\mathbf{G}\right)$, is zero under the following conditions:

- $\mathbf{P}$ is a fixed point.
- $\mathbf{l}$ is zero.
- $\mathring{\mathbf{P}}$ is parallel to $\mathring{\mathbf{P}}_\mathbf{G}$.
- $\mathbf{P}$ is equal to $\mathbf{P_G}$.

In particular then, when the point to which the momentum is referred is the centre of mass, the rate of change of momentum [4.2] becomes:

$$\boxed{\mathring{\mathbb{L}} == \mathbf{P_G} \wedge \mathring{\mathbf{l}} + \mathring{\mathbb{H}}_{\mathbf{P_G}}} \qquad \mathbf{8.17}$$

Newton's second law

For a system of bodies of momentum $\mathbb{L}$ acted upon by a system of forces $\mathbb{F}$, Newton's second law may be expressed as:

$$\boxed{\mathbb{F} == \overset{\circ}{\mathbb{L}}} \tag*{8.18}$$

This equation captures Newton's law in its most complete form, encapsulating both linear and angular terms. Substituting for $\mathbb{F}$ and $\overset{\circ}{\mathbb{L}}$ from equations 8.5 and 8.16 we have:

$$\boxed{\mathbf{P} \wedge \mathbf{f} + \mathbb{G}_{\mathbf{P}} == \overset{\circ}{\mathbf{P}} \wedge \mathbf{l} + \mathbf{P} \wedge \overset{\circ}{\mathbf{l}} + \overset{\circ}{\mathbb{H}}_{\mathbf{P}}} \tag*{8.19}$$

By equating the bound vector terms of [8.19] we obtain the vector equation:

$$\mathbf{f} == \overset{\circ}{\mathbf{l}}$$

By equating the bivector terms of [8.19] we obtain the bivector equation:

$$\mathbb{G}_{\mathbf{P}} == \overset{\circ}{\mathbf{P}} \wedge \mathbf{l} + \overset{\circ}{\mathbb{H}}_{\mathbf{P}}$$

In a metric 3-plane, it is the vector complement of this bivector equation which is usually given as the moment condition.

$$\overrightarrow{\mathbb{G}_{\mathbf{P}}} == \overset{\circ}{\mathbf{P}} \times \mathbf{l} + \overrightarrow{\overset{\circ}{\mathbb{H}}_{\mathbf{P}}}$$

If $\mathbf{P}$ *is a fixed point* so that $\overset{\circ}{\mathbf{P}}$ is zero, Newton's law [8.18] is equivalent to the pair of equations:

$$\boxed{\begin{gathered}\mathbf{f} == \overset{\circ}{\mathbf{l}} \\ \mathbb{G}_{\mathbf{P}} == \overset{\circ}{\mathbb{H}}_{\mathbf{P}}\end{gathered}} \tag*{8.20}$$

8.5 The Angular Velocity of a Rigid Body

To be completed.

8.6 The Momentum of a Rigid Body

To be completed.

8.7 The Velocity of a Rigid Body

`To be completed.`

8.8 The Complementary Velocity of a Rigid Body

`To be completed.`

8.9 The Infinitesimal Displacement of a Rigid Body

`To be completed.`

8.10 Work, Power and Kinetic Energy

`To be completed.`