7 Exploring Screw Algebra

7.1 Introduction

In Chapter 8: Exploring Mechanics, we will see that systems of forces and momenta, and the velocity and infinitesimal displacement of a rigid body may be represented by the sum of a bound vector and a bivector. We have already noted in Chapter 1 that a single force is better represented by a bound vector than by a (free) vector. Systems of forces are then better represented by sums of bound vectors; and a sum of bound vectors may always be reduced to the sum of a single bound vector and a single bivector.

We call the sum of a bound vector and a bivector a 2-*entity*. These geometric entities are therefore worth exploring for their ability to represent the principal physical entities of mechanics.

In this chapter we begin by establishing some properties of 2-entities in an *n*-plane, and then show how, in a 3-plane, they take on a particularly symmetrical and potent form. This form, a 2-entity in a 3-plane, is called a *screw*. Since it is in the 3-plane that we wish to explore 3-dimensional mechanics we then explore the properties of screws in more detail.

The objective of this chapter then, is to lay the algebraic and geometric foundations for the chapter on mechanics to follow.

Historical Note

The classic text on screws and their application to mechanics is by Sir Robert Stawell Ball: *A Treatise on the Theory of Screws* (1900). Ball was aware of Grassmann's work as he explains in the Biographical Notes to this text in a comment on the *Ausdehnungslehre* of 1862.

> This remarkable work, a development of an earlier volume (1844), by the same author, contains much that is of instruction and interest in connection with the present theory.
>
> ... Here we have a very general theory, which includes screw coordinates as a particular case.

The principal proponent of screw theory from a Grassmannian viewpoint was Edward Wyllys Hyde. In 1888 he wrote a paper entitled 'The Directional Theory of Screws' on which Ball comments, again in the Biographical Notes to *A Treatise on the Theory of Screws*.

> The author writes: "I shall define a screw to be the sum of a point-vector and a plane-vector perpendicular to it, the former being a directed and posited line, the latter the product of two vectors, hence a directed but not posited plane." Prof. Hyde proves by his [sic] calculus many of the fundamental theorems in the present theory in a very concise manner.

7.2 A Canonical Form for a 2-Entity

The canonical form

The most general 2-entity in a bound vector space may always be written as the sum of a bound vector and a bivector.

$$\mathbf{S} == \mathbf{P} \wedge \alpha + \beta$$

Here, $\mathbf{P}$ is a point, α a vector and β a bivector. Remember that only in vector 1, 2 and 3-spaces are bivectors necessarily simple. In what follows we will show that in a metric space the point $\mathbf{P}$ may always be chosen in such a way that *the bivector* β *is orthogonal to the vector* α. This property, when specialised to three-dimensional space, is important for the theory of screws to be developed in the rest of the chapter. We show it as follows:

To the above equation, add and subtract the bound vector $\mathbf{P}^* \wedge \alpha$ such that $(\mathbf{P} - \mathbf{P}^*) \ominus \alpha == 0$, giving:

$$\mathbf{S} == \mathbf{P}^* \wedge \alpha + \beta^*$$

$$\beta^* == \beta + (P - P^*) \wedge \alpha$$

We want to choose P^* such that β^* is orthogonal to α, that is:

$$(\beta + (P - P^*) \wedge \alpha) \ominus \alpha == 0$$

Expanding the left-hand side of this equation gives:

$$\beta \ominus \alpha + (\alpha \ominus (P - P^*))\, \alpha - (\alpha \ominus \alpha)\, (P - P^*) == 0$$

But from our first condition on the choice of P^* (that is, $(P - P^*) \ominus \alpha == 0$) we see that the second term is zero giving:

$$P^* == P - \frac{\beta \ominus \alpha}{\alpha \ominus \alpha}$$

whence β^* may be expressed as:

$$\beta^* == \beta + \left(\frac{\beta \ominus \alpha}{\alpha \ominus \alpha}\right) \wedge \alpha$$

Finally, substituting for P^* and β^* in the expression for S above gives the *canonical form* for the 2-entity S, in which the new bivector component is now orthogonal to the vector α. The bound vector component defines a line called the *central axis* of S.

$$\boxed{S == \left(P - \frac{\beta \ominus \alpha}{\alpha \ominus \alpha}\right) \wedge \alpha + \left(\beta + \left(\frac{\beta \ominus \alpha}{\alpha \ominus \alpha}\right) \wedge \alpha\right)} \qquad 7.1$$

Canonical forms in an *n*-plane

Canonical forms in $\mathbb{P}_1$

In a bound vector space of one dimension, that is a 1-plane, there are points, vectors, and bound vectors. There are no bivectors. Hence every 2-entity is of the form $S == P \wedge \alpha$ and is therefore in some sense already in its canonical form.

Canonical forms in $\mathbb{P}_2$

In a bound vector space of two dimensions (the plane), every bound element can be expressed as a bound vector.

To see this we note that any bivector is simple and can be expressed using the vector of the bound vector as one of its factors. For example, if P is a point and α, β_1, β_2, are vectors, any bound element in the plane can be expressed in the form:

$$P \wedge \alpha + \beta_1 \wedge \beta_2$$

But because any bivector in the plane can be expressed as a scalar factor times any other bivector, we can also write:

$$\beta_1 \wedge \beta_2 == \kappa \wedge \alpha$$

so that the bound element may now be written as the bound vector:

$$\mathbf{P}\wedge\alpha+\beta_1\wedge\beta_2 == \mathbf{P}\wedge\alpha+\kappa\wedge\alpha == (\mathbf{P}+\kappa)\wedge\alpha == \mathbf{P}^*\wedge\alpha$$

We can use *GrassmannAlgebra* to verify that formula 7.1 gives this result in a 2-space. First we declare the 2-space, and write α and β in terms of basis elements:

$$\mathbb{P}_2;\ \alpha = \mathbf{a}\,\mathbf{e}_1+\mathbf{b}\,\mathbf{e}_2;\ \beta = \mathbf{c}\,\mathbf{e}_1\wedge\mathbf{e}_2;$$

For the canonical expression [7.1] above we wish to show that the bivector term is zero. We do this by converting the interior products to scalar products using `ToScalarProducts`.

$$\texttt{Simplify}\left[\texttt{ToScalarProducts}\left[\beta+\left(\frac{\beta\ominus\alpha}{\alpha\ominus\alpha}\right)\wedge\alpha\right]\right]$$

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0
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In sum: A bound 2-element in the plane, $\mathbf{P}\wedge\alpha+\beta$, may always be expressed as a bound vector:

$$\boxed{\mathbf{S} == \mathbf{P}\wedge\alpha+\beta == \left(\mathbf{P}-\frac{\beta\ominus\alpha}{\alpha\ominus\alpha}\right)\wedge\alpha} \qquad \mathbf{7.2}$$

■ Canonical forms in $\mathbb{P}_3$

In a bound vector space of three dimensions, that is a 3-plane, every 2-element can be expressed as the sum of a bound vector and a bivector orthogonal to the vector of the bound vector. (The bivector is necessarily simple, because all bivectors in a 3-space are simple.)

Such canonical forms are called *screws*, and will be discussed in more detail in the sections to follow.

Creating 2-entities

A 2-entity may be created by applying the *GrassmannAlgebra* function `CreateElement`. For example in 3-dimensional space we create a 2-entity based on the symbol **s** by entering:

$$\mathbb{P}_3;\ \mathbf{S} = \texttt{CreateElement}\left[\underset{2}{\mathbf{s}}\right]$$

$$s_1\,\mathbb{O}\wedge e_1+s_2\,\mathbb{O}\wedge e_2+s_3\,\mathbb{O}\wedge e_3+s_4\,e_1\wedge e_2+s_5\,e_1\wedge e_3+s_6\,e_2\wedge e_3$$

Note that `CreateElement` automatically declares the generated symbols $\mathbf{s_i}$ as scalars by adding the pattern **s_**. We can confirm this by entering `Scalars`.

Scalars

$$\{a,\ b,\ c,\ d,\ e,\ f,\ g,\ h,\ \mathbb{k},\ (\_\ominus\_)\,?\,\texttt{InnerProductQ},\ s\_,\ \underset{0}{\_}\}$$

To explicitly factor out the origin and express the 2-element in the form $\mathbf{S} == \mathbf{P}\wedge\alpha+\beta$, we can use the *GrassmannAlgebra* function `GrassmannSimplify`.

$$\mathcal{G}[\mathbf{S}]$$

$$\mathbb{O}\wedge(e_1\,s_1+e_2\,s_2+e_3\,s_3)+s_4\,e_1\wedge e_2+s_5\,e_1\wedge e_3+s_6\,e_2\wedge e_3$$

7.3 The Complement of 2-Entity

Complements in an *n*-plane

In this section an expression will be developed for the complement of a 2-entity in an *n*-plane with a metric. In an *n*-plane, the complement of the sum of a bound vector and a bivector is the sum of a bound (*n*-2)-vector and an (*n*-1)-vector. It is of pivotal consequence for the theory of mechanics that such a complement in a 3-plane (the usual three-dimensional space) is itself the sum of a bound vector and a bivector. Geometrically, a quantity and its complement have the same measure but are orthogonal. In addition, for a 2-element (such as the sum of a bound vector and a bivector) in a 3-plane, the complement of the complement of an entity is the entity itself. These results will find application throughout Chapter 8: Exploring Mechanics.

The metric we choose to explore is the hybrid metric $\mathbf{G}_{ij}$ defined in [5.33], in which the origin is orthogonal to all vectors, but otherwise the vector space metric is arbitrary.

The complement referred to the origin

Consider the general sum of a bound vector and a bivector expressed in its simplest form referred to the origin. "Referring" an element to the origin means expressing its bound component as bound through the origin, rather than through some other more general point.

$$\mathbf{X}==\mathbb{O}\wedge\alpha+\beta$$

The complement of $\mathbf{X}$ is then:

$$\overline{\mathbf{X}}==\overline{\mathbb{O}\wedge\alpha}+\overline{\beta}$$

which, by formulae 5.41 and 5.43 derived in Chapter 5, gives:

$$\overline{\mathbf{X}}==\mathbb{O}\wedge\vec{\beta}+\vec{\alpha}$$

$$\boxed{\mathbf{X}==\mathbb{O}\wedge\alpha+\beta\quad\Longleftrightarrow\quad\overline{\mathbf{X}}==\mathbb{O}\wedge\vec{\beta}+\vec{\alpha}}\qquad\mathbf{7.3}$$

It may be seen that the effect of taking the complement of $\mathbf{X}$ when it is in a form referred to the origin is equivalent to interchanging the vector α and the bivector β whilst taking their vector space complements. This means that the vector that was bound through the origin becomes a free (*n*–1)-vector, whilst the bivector that was free becomes an (*n*–2)-vector bound through the origin.

The complement referred to a general point

Let $\mathbf{X} == \mathbb{O}\wedge\alpha + \beta$ and $\mathbf{P} == \mathbb{O} + \nu$. We can refer $\mathbf{X}$ to the point $\mathbf{P}$ by adding and subtracting $\nu\wedge\alpha$ to and from the above expression for $\mathbf{X}$.

$$\mathbf{X} == \mathbb{O}\wedge\alpha + \nu\wedge\alpha + \beta - \nu\wedge\alpha == (\mathbb{O} + \nu)\wedge\alpha + (\beta - \nu\wedge\alpha)$$

Or, equivalently:

$$\mathbf{X} == \mathbf{P}\wedge\alpha + \beta_{\mathbf{p}} \qquad\qquad \beta_{\mathbf{p}} == \beta - \nu\wedge\alpha$$

By manipulating the complement $\overline{\mathbf{P}\wedge\alpha}$, we can write it in the alternative form $-\overline{\alpha}\ominus\mathbf{P}$.

$$\overline{\mathbf{P}\wedge\alpha} == -\overline{\alpha\wedge\mathbf{P}} == -\overline{\alpha}\vee\overline{\mathbf{P}} == -\overline{\alpha}\ominus\mathbf{P}$$

Further, from formula 5.41, we have that:

$$-\overline{\alpha}\ominus\mathbf{P} == (\mathbb{O}\wedge\vec{\alpha})\ominus\mathbf{P}$$

$$\overline{\beta_{\mathbf{p}}} = \mathbb{O}\wedge\overrightarrow{\beta_{\mathbf{p}}}$$

So that the relationship between $\mathbf{X}$ and its complement $\overline{\mathbf{X}}$, can finally be written:

$$\boxed{\mathbf{X} == \mathbf{P}\wedge\alpha + \beta_{\mathbf{p}} \quad\Longleftrightarrow\quad \overline{\mathbf{X}} == \mathbb{O}\wedge\overrightarrow{\beta_{\mathbf{p}}} + (\mathbb{O}\wedge\vec{\alpha})\ominus\mathbf{P}} \qquad 7.4$$

Remember, this formula is valid for the hybrid metric [5.33] in an *n*-plane of arbitrary dimension. We explore its application to 3-planes in the section below.

7.4 The Screw

The definition of a screw

A screw is the *canonical form of a 2-entity in a three-plane*, and may always be written in the form:

$$\boxed{\mathbf{S} == \mathbf{P}\wedge\alpha + \mathbf{s}\,\vec{\alpha}} \qquad 7.5$$

where:

α	is the *vector* of the screw
$\mathbf{P}\wedge\alpha$	is the *central axis* of the screw
$\mathbf{s}$	is the *pitch* of the screw
$\vec{\alpha}$	is the *bivector* of the screw

Remember that $\vec{\alpha}$ is the (free) complement of the vector α in the three-plane, and hence is a simple bivector.

The unit screw

Let the scalar $\mathbf{a}$ denote the magnitude of the vector $\boldsymbol{\alpha}$ of the screw, so that $\alpha == \mathbf{a}\,\hat{\alpha}$. A unit screw $\hat{\mathbf{S}}$ may then be defined by $\mathbf{S} == \mathbf{a}\,\hat{\mathbf{S}}$, and written as:

$$\boxed{\hat{\mathbf{S}} = \mathbf{P} \wedge \hat{\alpha} + \mathbf{s}\,\vec{\hat{\alpha}}} \qquad 7.6$$

Note that the unit screw does not have unit magnitude. The magnitude of a screw will be discussed in Section 7.5.

The pitch of a screw

An explicit formula for the pitch $\mathbf{s}$ of a screw is obtained by taking the exterior product of the screw expression with its vector $\boldsymbol{\alpha}$.

$$\mathbf{S} \wedge \alpha == (\mathbf{P} \wedge \alpha + \mathbf{s}\,\vec{\alpha}) \wedge \alpha$$

The first term in the expansion of the right hand side is zero, leaving:

$$\mathbf{S} \wedge \alpha == \mathbf{s}\,\vec{\alpha} \wedge \alpha$$

Further, by taking the free complement of this expression and invoking the definition of the interior product we get:

$$\overrightarrow{\mathbf{S} \wedge \alpha} == \mathbf{s}\,\overrightarrow{\vec{\alpha} \wedge \alpha} == \mathbf{s}\,\alpha \vee \vec{\alpha} == \mathbf{s}\,\alpha \ominus \alpha$$

Dividing through by the square of the magnitude of $\boldsymbol{\alpha}$ gives:

$$\overrightarrow{\hat{\mathbf{S}} \wedge \hat{\alpha}} == \mathbf{s}\,\hat{\alpha} \ominus \hat{\alpha} == \mathbf{s}$$

In sum: We can obtain the pitch of a screw from the any of the following formulae:

$$\boxed{\mathbf{s} == \frac{\mathbf{S} \wedge \alpha}{\vec{\alpha} \wedge \alpha} == \frac{\overrightarrow{\mathbf{S} \wedge \alpha}}{\alpha \ominus \alpha} == \overrightarrow{\hat{\mathbf{S}} \wedge \hat{\alpha}}} \qquad 7.7$$

The central axis of a screw

An explicit formula for the central axis $\mathbf{P} \wedge \alpha$ of a screw is obtained by taking the interior product of the screw expression with its vector $\boldsymbol{\alpha}$.

$$\mathbf{S} \ominus \alpha == (\mathbf{P} \wedge \alpha + \mathbf{s}\,\vec{\alpha}) \ominus \alpha$$

The second term in the expansion of the right-hand side is zero (since an element is always orthogonal to its complement), leaving:

$$\mathbf{S}\ominus\alpha == (\mathbf{P}\wedge\alpha)\ominus\alpha$$

The right-hand side of this equation may be expanded using the Interior Common Factor Theorem to give:

$$\mathbf{S}\ominus\alpha == (\alpha\ominus\mathbf{P})\,\alpha - (\alpha\ominus\alpha)\,\mathbf{P}$$

By taking the exterior product of this expression with α, we eliminate the first term on the right-hand side to get:

$$(\mathbf{S}\ominus\alpha)\wedge\alpha == -(\alpha\ominus\alpha)\,(\mathbf{P}\wedge\alpha)$$

By dividing through by the square of the magnitude of α, we can express this in terms of unit elements.

$$(\mathbf{S}\ominus\hat{\alpha})\wedge\hat{\alpha} == -(\mathbf{P}\wedge\alpha)$$

In sum: We can obtain the central axis $\mathbf{P}\wedge\alpha$ of a screw from any of the following formulae:

$$\boxed{\mathbf{P}\wedge\alpha == -\frac{(\mathbf{S}\ominus\alpha)\wedge\alpha}{\alpha\ominus\alpha} == -(\mathbf{S}\ominus\hat{\alpha})\wedge\hat{\alpha} == \hat{\alpha}\wedge(\mathbf{S}\ominus\hat{\alpha})} \qquad \mathbf{7.8}$$

Orthogonal decomposition of a screw

By taking the expression for a screw and substituting the expressions derived in [7.7] for the pitch and [7.8] for the central axis we obtain:

$$\mathbf{S} == -(\mathbf{S}\ominus\hat{\alpha})\wedge\hat{\alpha} + \frac{\overrightarrow{\mathbf{S}\wedge\alpha}}{\alpha\ominus\alpha}\,\vec{\alpha}$$

In order to transform the second term into the form we want, we note first that $\overrightarrow{\mathbf{S}\wedge\alpha}$ is a scalar. So that we can write:

$$\overrightarrow{\mathbf{S}\wedge\alpha}\ \vec{\alpha} == \overrightarrow{\mathbf{S}\wedge\alpha}\wedge\vec{\alpha} == \overrightarrow{\overrightarrow{\mathbf{S}\wedge\alpha}\wedge\alpha} == (\mathbf{S}\wedge\alpha)\ominus\alpha$$

Hence $\mathbf{S}$ can be written as:

$$\boxed{\mathbf{S} == -(\mathbf{S}\ominus\hat{\alpha})\wedge\hat{\alpha} + (\mathbf{S}\wedge\hat{\alpha})\ominus\hat{\alpha}} \qquad \mathbf{7.9}$$

This type of decomposition is in fact valid for any *m*-element $\mathbf{S}$ and 1-element α, as we have shown in Chapter 6, formula 6.68. The central axis is the term $-(\mathbf{S}\ominus\hat{\alpha})\wedge\hat{\alpha}$ which is the component of $\mathbf{S}$ parallel to α, and the term $(\mathbf{S}\wedge\hat{\alpha})\ominus\hat{\alpha}$ is $s\,\vec{\alpha}$, the component of $\mathbf{S}$ orthogonal to α.

7.5 The Algebra of Screws

To be completed

7.6 Computing with Screws

To be completed