

# Table of Contents

---

## 1 Introduction

---

### 1.1 Background

- The mathematical representation of physical entities
- The central concept of the *Ausdehnungslehre*
- Comparison with the vector and tensor algebras
- Algebraicizing the notion of linear dependence
- Grassmann algebra as a geometric calculus

### 1.2 The Exterior Product

- The anti-symmetry of the exterior product
- Exterior products of vectors in a three-dimensional space
- Terminology: elements and entities
- The grade of an element
- Interchanging the order of the factors in an exterior product
- A brief summary of the properties of the exterior product

### 1.3 The Regressive Product

- The regressive product as a dual product to the exterior product
- Unions and intersections of spaces
- A brief summary of the properties of the regressive product
- The Common Factor Axiom
- The intersection of two bivectors in a three-dimensional space

### 1.4 Geometric Interpretations

- Points and vectors
- Sums and differences of points
- Determining a mass-centre
- Lines and planes
- The intersection of two lines

### 1.5 The Complement

- The complement as a correspondence between spaces
- The Euclidean complement
- The complement of a complement
- The Complement Axiom

### 1.6 The Interior Product

- The definition of the interior product
- Inner products and scalar products
- Calculating interior products
- Expanding interior products
- The interior product of a bivector and a vector

**1.7 Exploring Screw Algebra**

To be completed

**1.8 Exploring Mechanics**

To be completed

**1.9 Exploring Grassmann Algebra**

To be completed

**1.10 Exploring the Generalized Product**

To be completed

**1.11 Exploring Hypercomplex Algebra**

To be completed

**1.12 Exploring Clifford Algebra**

To be completed

**1.13 Exploring Grassmann Matrix Algebra**

To be completed

## 2 The Exterior Product

---

**2.1 Introduction****2.2 The Exterior Product**

Basic properties of the exterior product

✧ Setting up scalars in *GrassmannAlgebra*

✧ Entering exterior products

**2.3 Exterior Linear Spaces**

Constructing  $m$ -elements

Spaces and congruence

The associativity of the exterior product

✧ Calculating exterior products

**2.4 Axioms for Exterior Linear Spaces**

Summary of axioms

Grassmann algebras

On the nature of scalar multiplication

✧ Factoring scalars

✧ Grassmann expressions

✧ Calculating the grade of a Grassmann expression

**2.5 Bases**

Bases for exterior linear spaces

✧ Setting up a basis in *GrassmannAlgebra*

✧ Generating bases of exterior linear spaces

✧ Tables and palettes of basis elements

Standard ordering

Indexing basis elements of exterior linear spaces

**2.6 Cobases**

- Definition of a cobasis
- The cobasis of unity
- ✧ Tables and palettes of cobasis elements
- The cobasis of a cobasis

**2.7 Determinants**

- Determinants from exterior products
- Properties of determinants
- The Laplace expansion technique
- ✧ Calculating determinants

**2.8 Cofactors**

- Cofactors from exterior products
- The Laplace expansion in cofactor form
- ✧ Calculation of determinants using minors and cofactors
- Transformations of cobases

**2.9 Solution of Linear Equations**

- Grassmann's approach to solving linear equations
- ✧ Example solution

**2.10 Simplicity**

- The concept of simplicity
- All  $(n-1)$ -elements are simple
- ✧ Conditions for simplicity of a 2-element in a 4-space
- ✧ Conditions for simplicity of a 2-element in a 5-space

**2.11 Exterior Division**

- The definition of an exterior quotient
- Division by a 1-element
- Division by a  $k$ -element

## 3 The Regressive Product

---

**3.1 Introduction****3.2 Duality**

- The notion of duality
- Examples: Obtaining the dual of an axiom
- Summary: The duality transformation algorithm

**3.3 Properties of the Regressive Product**

- Axioms for the regressive product
- The unit  $n$ -element
- The inverse of an  $n$ -element

**3.4 The Duality Principle**

- The dual of a dual
- The Duality Principle
- ✧ Using the *GrassmannAlgebra* function Dual

**3.5 The Common Factor Axiom**

- Motivation
- The Common Factor Axiom
- Extension of the Common Factor Axiom to general elements
- Special cases of the Common Factor Axiom
- ✧ Application of the Common Factor Axiom

**3.6 The Common Factor Theorem**

- Development of the Common Factor Theorem
- Example: The decomposition of a 1-element
- ✧ Example: Applying the Common Factor Theorem
- ✧ Automating the application of the Common Factor Theorem

**3.7 The Regressive Product of Simple Elements**

- The regressive product of simple elements
- ✧ The regressive product of  $(n-1)$ -elements
- The cobasis form of the Common Factor Axiom
- The regressive product of cobasis elements

**3.8 Factorization of Simple Elements**

- Factorization using the regressive product
- ✧ Factorizing elements expressed in terms of basis elements
- The factorization algorithm
- Factorization of  $(n-1)$ -elements
- ✧ Obtaining a factorization of a simple  $m$ -element
- ✧ Determining if an element is simple

**3.9 Product Formulae for Regressive Products**

- The Product Formula
- The General Product Formula
- ✧ Exploring the General Product Formula
- Decomposition formulae
- Expressing an element in terms of another basis
- Product formulae leading to scalar results

## 4 Geometric Interpretations

---

**4.1 Introduction****4.2 Geometrically Interpreted 1-elements**

- Vectors
- Points
- ✧ A shorthand for declaring standard bases
- ✧ Example: Calculation of the centre of mass

**4.3 Geometrically Interpreted 2-elements**

- Simple geometrically interpreted 2-elements
- The bivector
- The bound vector
- ✧ Sums of bound vectors

**4.4 Geometrically Interpreted  $m$ -elements**

- Types of geometrically interpreted  $m$ -elements
- The  $m$ -vector
- The bound  $m$ -vector
- Bound simple  $m$ -vectors expressed by points

**4.5 Decomposition into Components**

- The shadow
- Decomposition in a 2-space
- Decomposition in a 3-space
- Decomposition in a 4-space
- Decomposition of a point or vector in an  $n$ -space

**4.6 Geometrically Interpreted Spaces**

- Vector and point spaces
- Coordinate spaces
- Geometric dependence
- Geometric duality

**4.7  $m$ -planes**

- $m$ -planes defined by points
- $m$ -planes defined by  $m$ -vectors
- $m$ -planes as exterior quotients
- The operator  $\partial$

**4.8 Line Coordinates**

- ✧ Lines in a plane
- ✧ Lines in a 3-plane
- Lines in a 4-plane
- Lines in an  $m$ -plane

**4.9 Plane Coordinates**

- ✧ Planes in a 3-plane
- Planes in a 4-plane
- Planes in an  $m$ -plane

**4.10 Calculation of Intersections**

- ✧ The intersection of two lines in a plane
- ✧ The intersection of a line and a plane in a 3-plane
- ✧ The intersection of two planes in a 3-plane
- ✧ Example: The osculating plane to a curve

## 5 The Complement

---

**5.1 Introduction**

**5.2 Axioms for the Complement**

The grade of a complement  
 The linearity of the complement operation  
 The complement axiom  
 The complement of a complement axiom  
 The complement of unity

**5.3 Defining the Complement**

The complement of an  $m$ -element  
 The complement of a basis  $m$ -element  
 Defining the complement of a basis 1-element  
 Determining the value of  $\mathbb{k}$

**5.4 The Euclidean Complement**

Tabulating Euclidean complements of basis elements  
 Formulae for the Euclidean complement of basis elements

**5.5 Complementary Interlude**

Alternative forms for complements  
 Orthogonality  
 Visualizing the complement axiom  
 The regressive product in terms of complements  
 Relating exterior and regressive products

**5.6 The Complement of a Complement**

The complement of a cobasis element  
 The complement of the complement of a basis 1-element  
 The complement of the complement of a basis  $m$ -element  
 The complement of the complement of an  $m$ -element

**5.7 Working with Metrics**

⊗ The default metric  
 ⊗ Declaring a metric  
 ⊗ Declaring a general metric  
 ⊗ Calculating induced metrics  
 ⊗ The metric for a cobasis  
 ⊗ Creating tables of induced metrics  
 ⊗ Creating palettes of induced metrics

**5.8 Calculating Complements**

⊗ Entering a complement  
 ⊗ Converting to complement form  
 ⊗ Simplifying complements  
 ⊗ Creating tables and palettes of complements of basis elements

**5.9 Geometric Interpretations**

The Euclidean complement in a vector 2-space  
 ⊗ The Euclidean complement in a plane  
 ⊗ The Euclidean complement in a vector 3-space

**5.10 Complements in a vector subspace of a multiplane**

Metrics in a multiplane  
 The complement of an  $m$ -vector

The complement of an element bound through the origin  
 The complement of the complement of an  $m$ -vector  
 The complement of a bound element  
 ✧ Calculating with free complements  
 Example: The complement of a screw

### 5.11 Reciprocal Bases

Contravariant and covariant bases  
 The complement of a basis element  
 The complement of a cobasis element  
 The complement of a complement of a basis element  
 The exterior product of basis elements  
 The regressive product of basis elements  
 The complement of a simple element is simple

### 5.12 Summary

## 6 The Interior Product

---

### 6.1 Introduction

#### 6.2 Defining the Interior Product

Definition of the interior product  
 Implications of the regressive product axioms  
 Orthogonality  
 ✧ The interior product of a vector with a simple bivector

#### 6.3 Properties of the Interior Product

Implications of the Complement Axiom  
 Extended interior products  
 Interior products of elements of the same grade  
 The inner product  
 Example: Orthogonalizing a set of 1-elements

#### 6.4 The Interior Common Factor Theorem

The Interior Common Factor Formula  
 The Interior Common Factor Theorem  
 ✧ Examples of the Interior Common Factor Theorem  
 ✧ The list form of the Interior Common Factor Theorem

#### 6.5 The Inner Product

Inner products of general elements  
 ✧ Calculating inner products  
 Inner products of basis elements

#### 6.6 The Measure of an $m$ -element

The definition of measure  
 Unit elements  
 ✧ Calculating measures  
 ✧ The measure of free elements  
 The measure of bound elements

**6.7 The Induced Metric Tensor**

- ✧ Calculating induced metric tensors
- ✧ Using scalar products to construct induced metric tensors
- ✧ Displaying induced metric tensors as a matrix of matrices

**6.8 Product Formulae for Interior Products**

- Interior product formulae for 1-elements
- Interior product formulae for 1-elements from regressive product formulae
- Interior product formulae for  $p$ -elements from regressive product formulae

**6.9 The Cross Product**

- Defining a generalized cross product
- Cross products involving 1-elements
- Implications of the axioms for the cross product
- The cross product as a universal product
- Cross product formulae

**6.10 The Triangle Formulae**

- Triangle components
- The measure of the triangle components
- Equivalent forms for the triangle components

**6.11 Angle**

- Defining the angle between elements
- ✧ The angle between a vector and a bivector
- ✧ The angle between two bivectors
- ✧ The volume of a parallelepiped

**6.12 Projection**

- To be completed.

**6.13 Interior Products of Interpreted Elements**

- To be completed.

**6.14 The Closest Approach of Multiplanes**

- To be completed.

## **7 Exploring Screw Algebra**

---

**7.1 Introduction****7.2 A Canonical Form for a 2-Entity**

- The canonical form
- ✧ Canonical forms in an  $n$ -plane
- ✧ Creating 2-entities

**7.3 The Complement of 2-Entity**

- Complements in an  $n$ -plane
- The complement referred to the origin
- The complement referred to a general point

**7.4 The Screw**

- The definition of a screw

The unit screw  
The pitch of a screw  
The central axis of a screw  
Orthogonal decomposition of a screw

### **7.5 The Algebra of Screws**

To be completed

### **7.6 Computing with Screws**

To be completed

## **8 Exploring Mechanics**

---

### **8.1 Introduction**

### **8.2 Force**

Representing force  
Systems of forces  
Equilibrium  
Force in a metric 3-plane

### **8.3 Momentum**

The velocity of a particle  
Representing momentum  
The momentum of a system of particles  
The momentum of a system of bodies  
Linear momentum and the mass centre  
Momentum in a metric 3-plane

### **8.4 Newton's Law**

Rate of change of momentum  
Newton's second law

### **8.5 The Angular Velocity of a Rigid Body**

To be completed.

### **8.6 The Momentum of a Rigid Body**

To be completed.

### **8.7 The Velocity of a Rigid Body**

To be completed.

### **8.8 The Complementary Velocity of a Rigid Body**

To be completed.

### **8.9 The Infinitesimal Displacement of a Rigid Body**

To be completed.

### **8.10 Work, Power and Kinetic Energy**

To be completed.

## **9 Exploring Grassmann Algebra**

---

To be completed

## **10 Exploring the Generalized Product**

---

To be completed

## **11 Exploring Hypercomplex Algebra**

---

To be completed

## **12 Exploring Clifford Algebra**

---

To be completed

## **13 Exploring Grassmann Matrix Algebra**

---

To be completed

## **Guide to *GrassmannAlgebra***

---

To be completed

## **A Brief Biography of Grassmann**

---

To be completed

## **Notation**

---

To be completed

## **Glossary**

---

To be completed

## **Bibliography**

---

To be completed

## **Index**

---

To be completed