Case Study: A Passive Filter Network

Introduction

Origin of the problem

The problem to be analyzed is described in *The Principles of Design* by Nam P. Suh (Oxford University Press, 1990), and in a paper *Using Taguchi Methods to Apply the Axioms of Design* by Stephen F. Filippone in Robotics and Computer-Integrated Manufacturing (Vol 6, No 2, 1989).

Filippone used Taguchi's experimental design method which involves choosing many discrete values for an experimental array.

Objective

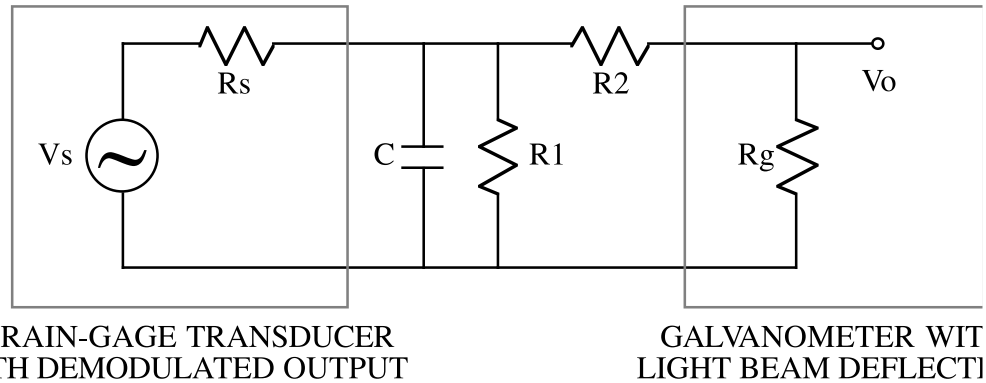
The objective of this case study is to demonstrate the method of Analytical Robustification.

We can use an analytical approach instead of an experimental one, because we can construct a model of the problem. We will see that this enables a much faster solution of higher accuracy leading to better insight.

We will use *Mathematica* to perform the symbolic computations for us.

Statement of the problem

A passive filter network is to be designed to measure the displacement signal generated by a strain gauge transducer. The network provides the interface between the strain gauge transducer/demodulator and the recording instrument with a galvanometer/light-beam deflection indicator. The network conditions the signal generated by a strain gauge transducer with demodulated output and measures the original displacement signal by filtering out the carrier frequency.



RC FILTER NETWORK

Nomenclature

C	Filter capacitance	Control parameter	Farad
R_1	Filter resistance	Control parameter	Ohm
R_2	Filter resistance	Control parameter	Ohm
R_g	Galvanometer resistance	Noise parameter	Ohm
R_s	Strain gauge resistance	Noise parameter	Ohm
V	Strain gauge voltage	Noise parameter	Volt
G	Galvanometer sensitivity	Noise parameter	Volts per inch
F	Filter cut-off frequency	Quality variable	Hertz
X	Galvanometer deflection	Quality variable	Inch
Ft	Target frequency	Target	Hertz
Xt	Target deflection	Target	Inch
ν_F	Variance of cut-off frequency	Calculated	$Hertz^2$
ν_X	Variance of galvanometer deflection	Calculated	$Inch^2$
k_1	Frequency variance weighting factor	Constant	$ / $Hertz^2$
k_2	Deflection variance weighting factor	Constant	$ / $Inch^2$
Q_T	Total quality loss		$

For simplicity we will denote the mean value of a variable by its symbol, as in the list above.

The quality variables

The filter cut-off frequency

◆ **The model**

$$F = \frac{\frac{1}{R_1} + \frac{1}{R_2 + R_g} + \frac{1}{R_s}}{2\, C\, \pi}$$

$$\frac{\frac{1}{R_1} + \frac{1}{R_2 + R_g} + \frac{1}{R_s}}{2\, C\, \pi}$$

◆ **The target frequency**

Target frequency = 6.84 Hertz

◆ **Put the frequency equal to its target value and solve for a design parameter (C)**

```
{{CRule}} = Solve[F == Ft, C]
```

$$\left\{\left\{C \to \frac{R_1\, R_2 + R_1\, R_g + R_1\, R_s + R_2\, R_s + R_g\, R_s}{2\, Ft\, \pi\, R_1\, (R_2 + R_g)\, R_s}\right\}\right\}$$

The galvanometer full scale deflection

◆ **The model**

$$X = \frac{R_1\, R_g\, V}{G\, (R_1\, R_s + (R_2 + R_g)\, (R_1 + R_s))}$$

$$\frac{V\, R_1\, R_g}{G\, (R_1\, R_s + (R_2 + R_g)\, (R_1 + R_s))}$$

◆ **The target deflection**

Target full-scale deflection = ±3 inches

◆ **Put the deflection equal to its target and solve for a design parameter (R_2)**

```
{{R2Rule}} = Solve[X == Xt, R2]
```

$$\left\{\left\{R_2 \to \frac{V R_1 R_g - G\, Xt\, R_1 R_g - G\, Xt\, R_1 R_s - G\, Xt\, R_g R_s}{G\, Xt\, (R_1 + R_s)}\right\}\right\}$$

Parameter values

◆ **Construct a list of substitutions for all the values we know.**

Standard deviations are taken to equate to one-third the semi-tolerance (reliable supplier).

Noise Parameters R_s, R_g, V and G are taken to have a tolerance of ± 0.15% of their mean.

Control Parameters R_1, R_2, and C are taken to have a tolerance of ± 5% of their mean.

$$A = \left\{\nu_s \to \left(\frac{0.0015}{3} R_s\right)^2,\ \nu_g \to \left(\frac{0.0015}{3} R_g\right)^2,\ \nu_v \to \left(\frac{0.0015}{3} V\right)^2,\ \nu_G \to \left(\frac{0.0015}{3} G\right)^2,\ \nu_1 \to \left(\frac{0.05}{3} R_1\right)^2,\ \nu_2 \to \left(\frac{0.05}{3} R_2\right)^2,\ \nu_c \to \left(\frac{0.05}{3} C\right)^2,\ Ft \to 6.84,\ Xt \to 3,\ R_s \to 120,\ R_g \to 98,\ V \to 0.015,\ G \to 0.00065758\right\};$$

The variances of the two quality variables

The variances of the quality variables are calculated using the first order approximation for the variance of a function of independent random variables.

The variance of the cut-off frequency

◆ **Write down the variance**

$$\nu_F = D[F, R_1]^2\, \nu_1 + D[F, R_2]^2\, \nu_2 + D[F, R_g]^2\, \nu_g + D[F, R_s]^2\, \nu_s + D[F, C]^2\, \nu_c$$

$$\frac{\nu_1}{4 C^2 \pi^2 R_1^4} + \frac{\nu_2}{4 C^2 \pi^2 (R_2 + R_g)^4} + \frac{\left(\frac{1}{R_1} + \frac{1}{R_2 + R_g} + \frac{1}{R_s}\right)^2 \nu_c}{4 C^4 \pi^2} + \frac{\nu_g}{4 C^2 \pi^2 (R_2 + R_g)^4} + \frac{\nu_s}{4 C^2 \pi^2 R_s^4}$$

◆ **Substituting in the nominal values and the variances**

```
νF1 = νF //. A
```

$$\frac{4.39762\times 10^{-13}}{C^2}+\frac{7.03619\times 10^{-6}}{C^2\,R_1^2}+\frac{0.000060818}{C^2\,(98+R_2)^4}+\frac{7.03619\times 10^{-6}\,R_2^2}{C^2\,(98+R_2)^4}+\frac{7.03619\times 10^{-6}\left(\frac{1}{120}+\frac{1}{R_1}+\frac{1}{98+R_2}\right)^2}{C^2}$$

◆ **Eliminating C and R_2 gives the variance of the frequency in terms of R_1 alone**

```
νF2 = νF1 /. CRule /. R2Rule //. A // Simplify
```

$$\frac{7353.78\,(5.35167-0.44293\,R_1+R_1^2)\,(24252.8+239.487\,R_1+R_1^2)}{R_1^2\,(89418.8+745.156\,R_1)^2}$$

The variance of the full-scale deflection

◆ **Write down the variance**

```
νX = D[X, R1]^2 ν1 + D[X, R2]^2 ν2 +
  D[X, Rg]^2 νg + D[X, Rs]^2 νs + D[X, V]^2 νv + D[X, G]^2 νG
```

$$\left(-\frac{V\,R_1\,R_g\,(R_2+R_g+R_s)}{G\,(R_1\,R_s+(R_2+R_g)\,(R_1+R_s))^2}+\frac{V\,R_g}{G\,(R_1\,R_s+(R_2+R_g)\,(R_1+R_s))}\right)^2 \nu_1+\frac{V^2\,R_1^2\,R_g^2\,(R_1+R_s)^2\,\nu_2}{G^2\,(R_1\,R_s+(R_2+R_g)\,(R_1+R_s))^4}+\left(-\frac{V\,R_1\,R_g\,(R_1+R_s)}{G\,(R_1\,R_s+(R_2+R_g)\,(R_1+R_s))^2}+\frac{V\,R_1}{G\,(R_1\,R_s+(R_2+R_g)\,(R_1+R_s))}\right)^2 \nu_g+\frac{V^2\,R_1^2\,R_g^2\,\nu_G}{G^4\,(R_1\,R_s+(R_2+R_g)\,(R_1+R_s))^2}+\frac{V^2\,R_1^2\,R_g^2\,(R_1+R_2+R_g)^2\,\nu_s}{G^2\,(R_1\,R_s+(R_2+R_g)\,(R_1+R_s))^4}+\frac{R_1^2\,R_g^2\,\nu_v}{G^2\,(R_1\,R_s+(R_2+R_g)\,(R_1+R_s))^2}$$

◆ **Substituting in the nominal values and the variances**

$$\nu_{X1} = \nu_X \text{ //. } A$$

$$\frac{1388.15\, R_1^2\, (120 + R_1)^2\, R_2^2}{(120\, R_1 + (120 + R_1)\, (98 + R_2))^4} + \frac{17990.4\, R_1^2\, (98 + R_1 + R_2)^2}{(120\, R_1 + (120 + R_1)\, (98 + R_2))^4} + \frac{2.49866\, R_1^2}{(120\, R_1 + (120 + R_1)\, (98 + R_2))^2} + 0.000277778\, R_1^2 \left(-\frac{2235.47\, R_1\, (218 + R_2)}{(120\, R_1 + (120 + R_1)\, (98 + R_2))^2} + \frac{2235.47}{120\, R_1 + (120 + R_1)\, (98 + R_2)}\right)^2 + 0.002401 \left(-\frac{2235.47\, R_1\, (120 + R_1)}{(120\, R_1 + (120 + R_1)\, (98 + R_2))^2} + \frac{22.8109\, R_1}{120\, R_1 + (120 + R_1)\, (98 + R_2)}\right)^2$$

◆ **Eliminating C and R_2 gives the variance of the frequency in terms of R_1 alone**

$$\nu_{X2} = \nu_{X1} \text{ /. CRule /. R2Rule //. A // Simplify}$$

$$\frac{0.00125745\, (250.358 - 20.1722\, R_1 + R_1^2)\, (28507.7 + 215.786\, R_1 + R_1^2)}{R_1^2\, (120 + R_1)^2}$$

Define an overall quality loss equal to the weighted sum of the variances

◆ **The quality loss**

The *quality loss* associated with a quality variable is defined as proportional to the variance of the quality variable under the condition that its mean is on target.

$$Q_F = k_F\, \nu_{F2} \qquad\qquad Q_X = k_X\, \nu_{X2}$$

The constants of proportionality k_F and k_X have units \$/hertz^2 and \$/inch^2.

◆ **The total quality loss**

The *total* quality loss is the sum of the losses associated with each quality variable

$$Q_T = Q_F + Q_X = k_F\, \nu_{F2} + k_X\, \nu_{X2}$$

◆ Estimating the relative quality losses

In order to minimize the total quality loss, we must make an assumption about the relative costs incurred by being off-target in the frequency compared to being off-target in the deflection.

Suppose we determine that the average quality loss incurred by being off-target by 1 hertz is the same as that incurred by being off-target by 0.1 inches, then

$$Q_F = k_F \times 1^2 = Q_X = k_X \times 0.1^2 \quad \text{implies that} \quad k_X = 100\, k_F \text{ and}$$

$$Q_T = Q_F + Q_X = k_F\, (\nu_{F2} + 100\, \nu_{X2})$$

We still do not know the value of k_F, but since it is a constant factor it does not affect the final optimized values (just the final scaling of the minimum quality loss). For simplicity we put it equal to 1.

$$Q_T = \nu_{F2} + 100\, \nu_{X2} \text{ // Simplify}$$

$$(77174.6\, (229.447 - 18.5268\, R_1 + R_1^2)\, (28195.6 + 218.279\, R_1 + R_1^2)\, (14400. + 240.\, R_1 + R_1^2)) \,/\, (R_1^2\, (120 + R_1)^2\, (89418.8 + 745.156\, R_1)^2)$$

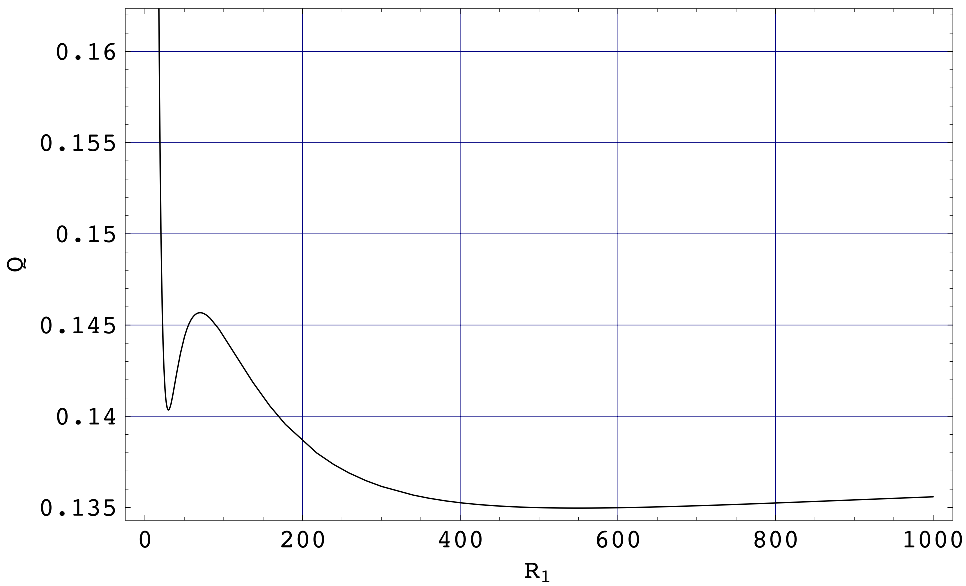
Plot of the total quality loss Q_T against R_1

(We cannot use subscripts in the Plot routine so we temporarily rewrite R_1 as R1)

$$Q_T = Q_T \text{ /. } R_1 \to \text{R1};$$

- **Plot over the range of interest**

```
Plot[Q_T, {R1, 10, 1000}, Frame → True,
 GridLines → Automatic, FrameLabel → {"R_1", "Q"}]
```




```
- Graphics -
```

- **Find the minimum numerically**

```
{Qmin, R1Rule} = FindMinimum[Q_T, {R1, 500}]
```

```
{0.134966, {R1 → 549.793}}
```

```
R1min = R1/.R1Rule
```

```
549.793
```

- **Substitute back to find the corresponding value of R_2**

```
R2min = R_2 /. R2Rule /. A /. R_1 → R1min
```

```
415.154
```

- **Substitute back to find the corresponding value of C**

```
Cmin = C /. CRule /. A /. {R_1 → R1min, R_2 → R2min}
```

```
0.000281568
```


◆ **Calculate the variance of the frequency and deflection at these values**

```
νFmin = νF2 /. R1 → R1min
```

0.0135158

```
νXmin = νX2 /. R1 → R1min
```

0.0012145

◆ **Check the values of the frequency and deflection to see if they are on target**

```
F /. A //. {R1 → R1min, R2 → R2min, C → Cmin}
```

6.84

```
X /. A //. {R1 → R1min, R2 → R2min, C → Cmin}
```

3.

◆ **Check the quality loss**

```
Qmin == νFmin + 100 νXmin
```

True

Summary of design resulting from minimization of variance

The plot of combined quality loss against one of the system parameters has shown that the solution is itself quite robust due to the small gradients near the minimum. Note however that there is another local minimum which is not robust, and indeed dangerous due to the high gradients surrounding it.

The final optimum results which produce the most robust design of this circuit are:

Q	**Quality Loss**	**0.135**	**$/Hz^2**
R_1	**Resistance**	**550**	**Ω**
R_2	**Resistance**	**415**	**Ω**
C	**Capacitance**	**282**	**μF**