Robust Design Problems

Problem 3.45 Sources of variation

Problem

Discuss the types of design parameters that can cause variation in the performance of a product.

Solution

The types of *basic* design parameters that can cause variation in the performance of a product are:
Material properties: for example, density, modulus of elasticity, Poisson's ratio, shear modulus, yield strength, ultimate tensile strength, bulk modulus, coefficient of thermal expansion, thermal conductivity, electrical conductivity.
Dimensional parameters: for example: length, diameter, angle.
System or environmental parameters: for example: applied load, ambient temperature, ambient pressure, humidity, applied voltage.

The designer will usually have control over the setting of the values of material properties and dimensions, but not over the system or environmental parameters.

There are also *derived* design parameters which are functions of basic design parameters: for example, spring stiffness, beam deflection, electrical resistance. These should be distinguished from *quality variables* which are also functions of design parameters, but are *outputs* rather than inputs of the design process.

Problem 3.46 Designing robust products

Problem

Write brief notes on the advantages and disadvantages of designing robust products.

Solution

Advantages of designing robust products are:
Quality: The product is more likely to continue to satisfy the customer over its intended life.
Reliability: The reliability of the product can be predicted and optimized.
Cost: Cost may be reduced by using lower tolerances and cheaper materials without compromising performance.
Maintainability: Service costs can be predicted and minimized.
Time-to-market: Product development, testing and ramp-up to production are reduced. The likelihood of recall is reduced.

Disadvantages of designing robust products are:
Failure-mode-analysis: Important failure modes need to be identified.
Analytical prototyping: Mathematical models of the quality variables involved in the failure modes need to be created.
Robustification: The actual robustification process needs to be undertaken, either by simulation or other means of reducing variance.

Problem 3.47 Terminology for robust design

Problem

Explain the terms *failure mode*, *control parameter*, *noise parameter*, *quality variable* and *critical parameter*.

Solution

Failure mode: A failure mode is any of the ways in which a product can fail to perform as intended. Failure modes are usually due to a quality variable being outside a target range.

Control parameter: A control parameter is a parameter over whose nominal value the designer has direct control. The nominal values of control parameters are therefore varied over a range of feasible values during the optimization or robustification process.

Noise parameter: A noise parameter is a design parameter over whose nominal value the designer has no direct control. The nominal values of noise parameters are usually set by system requirements, user input, or the environment.

Quality variable: A quality variable is a function of design parameters which, if kept within its target range, will ensure one aspect of the quality of the product. Conversely, if a quality variable is outside its target range, the product is considered to fail by the corresponding failure mode.

Critical parameter: A critical parameter is a design parameter or quality variable whose value is critical to the performance of the product, or which is difficult to maintain at its target value.

Problem 3.48 The concept of robustness

Problem

Explain in your own words the concept of robustness using diagram(s) where appropriate.

Solution

Outline:
Include the following points:
1. Failure modes must be able to be modeled by requiring their corresponding quality variables to remain on target.
2. The formulae for the nominal values of the quality variables must be non-linear.
3. Of all the combinations of control parameters which maintain the quality variable on target, the combination which minimizes the probability of failure gives the most robust design.

Problem 3.49 First order robustness problem

Problem

What is the first order robustness problem?

Solution

The first order robustness problem is:
For a given mean of the quality variable and variances of the design parameters, determine the means of the design parameters which minimize the variance of the quality variable.

In more detail the first order robustness problem is:
Suppose a quality variable z is expressed in terms of design parameters x and y by $z = \mathrm{g}(x, y)$.
The design parameters x and y are considered to be independent random variables whose probability distributions have means μ_x and μ_y and variances ν_x and ν_y.
The quality variable z is also a random variable with mean μ_z and variance ν_z (dependent on the distributions of x and y).
Suppose that to ensure that the product performs as intended, we need to keep the quality variable as close as possible to the target value τ_z despite the variation of x and y.
The first order robustness problem is to determine, for given values of the variances ν_x and ν_y, the values of μ_x and μ_y which minimize the variance ν_z while keeping μ_z on target τ_z.

It is called the first order problem because it contains a number of "first order" approximations as part of the solution process.Extension to any number of parameters follows *mutatis mutandis*.

Problem 3.50 Exploitation of non-linearity

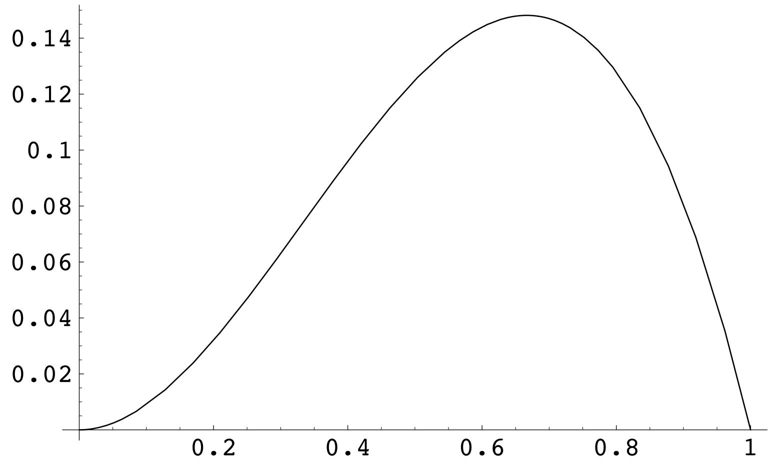
Problem

A quality variable **q** is related to its one major design parameter **x** by $\mathbf{q} = \mathbf{x}^2 \ (1 - \mathbf{x})$.
If the target value for **q** is 0.05, and the feasible range of **x** is $0.1 \le \mathbf{x} \le 1$, determine the value of **x** which will give the most robust design.
If **x** is uniformly distributed with support of 0.1 $(\mu_{\mathbf{x}} \pm 0.05)$, compare the variations in **q** due to the different possible settings of **x**.

Solution

```
q = x^2 (1 - x);
```

```
Plot[q, {x, 0, 1}]
```




```
- Graphics -
```

```
xx = NSolve[q == 0.05, x]
```

```
{{x → -0.203802}, {x → 0.259924}, {x → 0.943877}}
```

$x_1 = x$ /. xx[[2]]

0.259924

$x_2 = x$ /. xx[[3]]

0.943877

The most robust design is given by x_1 since the gradient at that position is less than that at x_2 .

Setting 1:

q11 = q /. x → (x_1 - 0.05)

0.0348172

q12 = q /. x → (x_1 + 0.05)

0.0662839

q1 = q12 - q11

0.0314667

Setting 2:

q21 = q /. x → (x_2 - 0.05)

0.0847938


```
q22 = q /. x → (x2 + 0.05)
```

0.00604801

```
q2 = q21 - q22
```

0.0787458

```
q1 / q2
```

0.399598

Thus the variation in **q** due to setting 1 is just 40% of the variation in **q** due to setting 2.

Problem 3.51 Derivation of average quality loss

Problem

Using Taguchi's assumption that the quality loss Q for an item is proportional to its deviation from the target value τ, derive an expression for the average quality loss of a large number of similar product whose quality variable is random distributed with mean μ and standard deviation σ.

Solution

The quality loss for one item is

$$Q_i = k\,(y_i - \tau)^2$$

The average quality loss is

$$\bar{Q} = \frac{\Sigma\, Q_i}{n} = k\,\frac{\Sigma\,(y_i - \tau)^2}{n}$$

Adding and subtracting μ gives

$$= k\,\frac{\Sigma\,((y_i - \mu) - (\tau - \mu))^2}{n}$$

$$= k\left(\frac{\Sigma\,(y_i - \mu)^2}{n} - 2\,(\tau - \mu)\,\frac{\Sigma\,(y_i - \mu)}{n} + \frac{\Sigma\,(\tau - \mu)^2}{n}\right)$$

The middle term is zero by definition

$$= k\,(\sigma_y^2 + (\tau - \mu)^2)$$

Problem 3.52 Example of average quality loss

Problem

It has been determined that a new model of car jack is returned by customers when it fails to operate smoothly due to the width of the square thread being less than 3.8 mm or greater than 4.2 mm. Although a jack retails for only $50, the total cost to society of a returned jack is estimated at $250. Calculate the average quality loss per jack in a production run in which the thread width is distributed with mean 4.1 mm and standard deviation 0.05 mm.
Comment on your result.

Solution

Using the formula derived above,

$$Q = k\,(\sigma^2 + (\tau - \mu)^2);$$

we can evaluate k from any point on the parabola

$$k = 250 / 0.2^2$$

```
6250.
```

Substituting the rest of the values gives

$$\bar{Q} = Q\ /.\ \{k \to 6250,\ \tau \to 4,\ \mu \to 4.1,\ \sigma \to 0.05\}$$

```
78.125
```

Comment: Thus, although the jack sells for just $50, the jack is incurring an average loss to society of $78 per jack! This is made up of the actual cost of returned jacks, and an estimate of the (harder-to-identify) customer losses due to the poor quality.

Problem 3.53 Robustification methodology

Problem

Explain in your own words the procedure you would adopt and the formulae you would use for robustifying a design.

Solution

Use as an outline the stages described in the algorithm of the section of the notes: First Order Analytical Robust Design.

Problem 3.54 Simple power model

Problem

Derive the formulae of the previous section. (That is, derive the robustification formulae for the simple power model.)

Solution

1. The model

$$z == a\, x^m\, y^n$$

2. Calculate the first order approximation to the mean

$$\mu_z == a\, \mu_x{}^m\, \mu_y{}^n$$

3. Calculate the first order approximation to the variance

$$\nu_z == \left[\frac{\partial z}{\partial x}\right]_\mu^2 \nu_x + \left[\frac{\partial z}{\partial y}\right]_\mu^2 \nu_y$$

4. Calculate the derivatives

$$\frac{\partial z}{\partial x} == m\, a\, x^{m-1}\, y^n == \frac{m\, z}{x} \qquad \frac{\partial z}{\partial y} == n\, a\, x^m\, y^{n-1} == \frac{n\, z}{y}$$

5. Evaluate the derivatives at their mean values

$$\left[\frac{\partial z}{\partial x}\right]_\mu == \frac{m\, \mu_z}{\mu_x} \qquad \left[\frac{\partial z}{\partial y}\right]_\mu == \frac{n\, \mu_z}{\mu_y}$$

6. Set the mean of the quality variable equal to the target value

$$\mu_z == \tau_z$$

7. Substitute into the variance formula

$$\nu_z == \tau_z^2 \left(\frac{m^2 \nu_x}{\mu_x^2} + \frac{n^2 \nu_y}{\mu_y^2} \right)$$

8. Choose one of the parameters as the adjustment parameter and solve for its mean

$$\mu_x == \left(\frac{\tau_z}{a \mu_y^n} \right)^{\frac{1}{m}}$$

9. Eliminate this mean from the variance formula

$$\nu_z == \tau_z^2 \left(\frac{m^2 \nu_x}{\left(\frac{\tau_z}{a \mu_y^n} \right)^{\frac{2}{m}}} + \frac{n^2 \nu_y}{\mu_y^2} \right)$$

10. Minimize the variance of z with respect to the mean of the remaining control parameter

$$\frac{\partial \nu_z}{\partial \mu_y} == \tau_z^2 \left(\frac{m^2 \nu_x}{\left(\frac{\tau_z}{a} \right)^{\frac{2}{m}}} \left(2 \frac{n}{m} \mu_y^{2 \frac{n}{m} - 1} \right) - 2 \frac{n^2 \nu_y}{\mu_y^3} \right)$$

11. Put the derivative equal to zero for a minimum

$$\frac{m^2 \nu_x}{\left(\frac{\tau_z}{a} \right)^{\frac{2}{m}}} \left(2 \frac{n}{m} \mu_y^{2 \frac{n}{m} - 1} \right) == 2 \frac{n^2 \nu_y}{\mu_y^3}$$

12. Solve for the mean of the remaining control parameter

$$\mu_y == \left(\left(\frac{\tau_z}{a}\right)\left(\frac{n}{m}\frac{\nu_y}{\nu_x}\right)^{\frac{m}{2}}\right)^{\frac{1}{m+n}}$$

13. Obtain the formula for the other control parameter by symmetry

$$\mu_x == \left(\left(\frac{\tau_z}{a}\right)\left(\frac{m}{n}\frac{\nu_x}{\nu_y}\right)^{\frac{n}{2}}\right)^{\frac{1}{m+n}}$$

14. Substitute for $\frac{\tau_z}{a}$

$$\mu_y == \left(\left(\frac{n}{m}\frac{\nu_y}{\nu_x}\right)^{\frac{m}{2}}\mu_x{}^m\,\mu_y{}^n\right)^{\frac{1}{m+n}}$$

15. Simplify this to get the relationship between the means and variances

$$\frac{\mu_y{}^2}{\mu_x{}^2} == \frac{n}{m}\frac{\nu_y}{\nu_x}$$

16. Put this in coefficient of variation form

$$m\,\hat{x}^2 == n\,\hat{y}^2$$

17. Substitute this result in the variance to find its minimum value

$$\nu_{z\,min} == m\,(m+n)\,\tau_z{}^2\,\hat{x}^2 == n\,(m+n)\,\tau_z{}^2\,\hat{y}^2$$

Problem 3.55 Stiffness of a cantilever

Problem

The stiffness of a cantilever spring has the model

$$k == \frac{Y\,W\,H^3}{4\,L^3}$$

Suppose that **Y** and **L** are constant parameters and that **W** and **H** are control parameters.
Using the results obtained from the simple power model above:
1. Determine the relationship between the coefficients of variation of the control parameters which will ensure that **k** has minimum variance.
2. Determine an expression for the best values of the control parameters.
3. Determine an expression for the minimum variance for **k**.

Solution

1. The relationship between $\hat{W}$ and $\hat{H}$ which will ensure minimum variance of k

$$\hat{W}^2 == 3\,\hat{H}^2$$

2. The best values of the control parameters

$$\mu_{W_{best}} == \left(\frac{\tau_k}{\left(\frac{\mu_Y}{4\,\mu_L{}^3}\right)}\left(\frac{1}{3}\,\frac{\nu_W}{\nu_H}\right)^{\frac{3}{2}}\right)^{\frac{1}{4}}$$

$$\mu_{H_{best}} == \left(\frac{\tau_k}{\left(\frac{\mu_Y}{4\,\mu_L{}^3}\right)}\left(3\,\frac{\nu_H}{\nu_W}\right)^{\frac{1}{2}}\right)^{\frac{1}{4}}$$

3. The minimum variance of k

The coefficient of variation of k is given by

$$\hat{k}^2 == \hat{Y}^2 + 9\,\hat{L}^2 + \hat{W}^2 + 9\,\hat{H}^2 == \hat{Y}^2 + 9\,\hat{L}^2 + 12\,\hat{H}^2$$

The minimum variance of k is then

$$\nu_{k_{min}} == \tau_k^2 \left(\frac{\nu_Y}{\mu_Y^2} + 9 \frac{\nu_L}{\mu_L^2} + 12 \frac{\nu_H}{\mu_H^2} \right)$$

Problem 3.56 Wooden cantilever

Problem

A flexible element is required with a given stiffness characteristic. The proposed design consists of a single cantilever sawn from wood of rectangular cross-section where the force-deflection characteristics of its tip are used for the spring.
The following data is given:
The target stiffness is τ_k = 0.25 N/mm
The control parameters are the width **W** and height **H** with tolerances ±3 mm.
The constant parameters are the Young's modulus of wood (Douglas Fir) **Y**: 10 000 ± 3000 N/mm^2 and the length **L** of the beam 1000 ± 3 mm.
Using the results obtained from the previous example
1. Design a cantilever by choosing values of **W** and **H** to keep the mean of **k** on target. Making reasonable assumptions for estimating the variances from the tolerances calculate the standard deviation of the stiffness.
2. Determine the cross-section that will minimize the variance of the stiffness.
3. Calculate the resulting minimum standard deviation of the stiffness, and compare it to your first design.
4. Estimate the material savings.

Solution

1. Initial cantilever design

The target stiffness is

$$\tau_k = 0.25\ N / mm$$

This target is achieved by the following values of the design parameters

$$\frac{Y\,W\,H^3}{4\,L^3} \ /.\ \{Y \to 10000,\ L \to 1000,\ W \to 100,\ H \to 10.\}$$

0.25

Suppose that a reliable and experienced manufacturer has been contracted. Standard deviations are then taken as one-third of the tolerances. We then substitute these into the formula for the variance derived in the previous problem.

$$\nu_k == \tau_k{}^2 \left(\frac{\nu_Y}{\mu_Y{}^2} + 9\,\frac{\nu_L}{\mu_L{}^2} + \frac{\nu_W}{\mu_W{}^2} + 9\,\frac{\nu_H}{\mu_H{}^2} \right) /. \{\tau_k \to 0.25,\ \mu_Y \to 10000,\ \mu_L \to 1000,\ \mu_W \to 100,\ \mu_H \to 10,\ \nu_Y \to 1000^2,\ \nu_L \to 1^2,\ \nu_W \to 1^2,\ \nu_H \to 1^2\}$$

$\nu_k == 0.00625681$

The standard deviation of the stiffness for the initial design is

$$\sigma_{k_{min}} == \sqrt{0.00625681}$$

$(\sigma_k)_{min} == 0.0791$

The tolerance range within which it is expected that the stiffness of this product will lie is then

$$k_{tol} == 3\,\sigma_{k_{min}} == 0.2373\ \text{N} / \text{mm}$$

The stiffness is then expected to be

$$k == 0,\ 25 \pm 0.24$$

This is clearly unacceptable!

2. The cross-section that will minimize the variance of the stiffness

Better values of **H** and **W** are found from the formulae derived in the previous problem.

$$\mu_{Wbest} == \left(\frac{\tau_k}{\left(\frac{\mu_Y}{4\,\mu_L{}^3}\right)} \left(\frac{1}{3}\,\frac{\nu_W}{\nu_H} \right)^{\frac{3}{2}} \right)^{\frac{1}{4}} /. \{\tau_k \to 0.25,\ \mu_Y \to 10000,\ \mu_L \to 1000,\ \mu_W \to 100,\ \mu_H \to 10,\ \nu_Y \to 1000^2,\ \nu_L \to 1^2,\ \nu_W \to 1^2,\ \nu_H \to 1^2\}$$

$\mu_{Wbest} == 11.7782$

$$\mu_{Hbest} == \left(\frac{\tau_k}{\left(\frac{\mu_Y}{4\,\mu_L{}^3}\right)} \left(3\,\frac{\nu_H}{\nu_W} \right)^{\frac{1}{2}} \right)^{\frac{1}{4}} /. \{\tau_k \to 0.25,\ \mu_Y \to 10000,\ \mu_L \to 1000,\ \mu_W \to 100,\ \mu_H \to 10,\ \nu_Y \to 1000^2,\ \nu_L \to 1^2,\ \nu_W \to 1^2,\ \nu_H \to 1^2\}$$

$\mu_{Hbest} == 20.4005$

Confirm that these values keep the mean stiffness on target

$$\frac{Y\,W\,H^3}{4\,L^3} /. \{Y \to 10000,\ L \to 1000,\ W \to 11.8,\ H \to 20.4\}$$

0.250445

Hence the cross-section that will minimize the variance of the stiffness is

$$W \times H == 11.8\,mm \times 20.4\,mm$$

The resulting minimum standard deviation of the stiffness

$$\nu_k == \tau_k^2 \left(\frac{\nu_Y}{\mu_Y^2} + 9\,\frac{\nu_L}{\mu_L^2} + \frac{\nu_W}{\mu_W^2} + 9\,\frac{\nu_H}{\mu_H^2} \right) /. \{\tau_k \to 0.25,\ \mu_Y \to 10000,\ \mu_L \to 1000,\ \mu_W \to 11.8,\ \mu_H \to 20.4,\ \nu_Y \to 1000^2,\ \nu_L \to 1^2,\ \nu_W \to 1^2,\ \nu_H \to 1^2\}$$

$$\nu_k == 0.00242607$$

$$\sigma_{k_{min}} == \sqrt{0.00242607}$$

$$(\sigma_k)_{min} == 0.0492552$$

The tolerance range within which it is expected that the stiffness of this new design will lie is then

$$k_{tol} == 3\,\sigma_{k_{min}} == 0.148\,N/mm$$

The new stiffness is then expected to be

$$k == 0,\ 25 \pm 0.15$$

Even though the variability of this design has been reduced to just 62% of that of the initial design, this result is unlikely to be acceptable either. However, since this is the minimum variance that can be obtained through robustification, further reduction in variance would need to be accomplished by reduction in modulus and dimensional tolerances.

Problem 3.57 Helical spring

Problem

A helical spring is to be designed for mass production with its deflection δ under force **F** to be as close as possible to a target specification of 5 mm.

The formula for the deflection of a helical spring is given by:

$$\delta = \frac{8\,F\,D^3\,N}{d^4\,G}$$

The data for the spring design as originally specified is given in the table:

"HELICAL SPRING"	Sym	Nominal	Tol	Unit	Type
"Coil Diameter"	D	20	± 3	mm	"Control"
"Number of Coils"	N	10	± 0.6	1	"Control"
"Applied Force"	F	50	± 9	N	"Noise"
"Wire Diameter"	d	3	± 0.06	mm	"Noise"
"Shear Modulus"	G	79000	± 1200	$\frac{N}{mm^2}$	"Noise"
"Deflection"	δ	5.00	□	mm	"Quality"

Consider that the springs are to be used in a military device, and hence will be manufactured under military standards.

1) For the values in the above table, estimate a specification tolerance for the deflection outside of which you would expect 64 per million of the springs to lie.

2) Using the formulae you have derived in Problem 1:
a) Determine the best nominal values for the control parameters for reducing the variation in the deflection. (Assume the tolerances remain fixed).
b) Determine the new specification tolerance for the deflection outside of which you would expect 32 per million of the springs to lie, and compare this to the original design.

3) Compare the cost of the material used between the original and the new design.

[Note carefully: These calculations *do not take in to account other important design constraints*, for example maximum stress. However, they do indicate the directions in which the values of design parameters may be moved to minimize their effect on variability.]

Solution

The nominal parameter values are

```
nominals = {D -> 20, N -> 10, F -> 50, d -> 3, G -> 79000};
```

The nominal deflection is

$$\frac{8\,F\,D^3\,N}{d^4\,G}\ \text{/. nominals // N}$$

```
5.00078
```

The standard deviations of the parameters are

```
stdevs = {sD -> 1, sN -> 0.2, sF -> 3, sd -> 0.02, sG -> 400};
```

$$\delta = \frac{8\,F\,D^3\,N}{d^4\,G}$$

The standard deviation of the deflection is

```
δhat2 = 9 (sD / D) ^2 + (sN / N) ^2 + (sF / F) ^2 +
    16 (sd / d) ^2 + (sG / G) ^2 /. nominals /. stdevs
```

0.0272367

$s\delta = 5\sqrt{\delta hat2}$

0.825178

64 per million translates to ±4 standard deviations:

```
tolδ = ±4 sδ
```

3.30071

Best nominal values

$$Dbest = \left(\sqrt{3}\ \frac{sD}{sN}\ \frac{\delta target}{\alpha}\right)^{\frac{1}{4}}$$

$$Nbest = \left(\left(\frac{1}{\sqrt{3}}\ \frac{sN}{sD}\right)^3\ \frac{\delta target}{\alpha}\right)^{\frac{1}{4}}$$

$$Db = Dbest\ /.\ stdevs\ /.\ \left\{\delta target \to 5,\ \alpha \to \frac{8\,F}{d^4\,G}\right\}\ /.\ nominals$$

28.8495

$$Nb = Nbest\ /.\ stdevs\ /.\ \left\{\delta target \to 5,\ \alpha \to \frac{8\,F}{d^4\,G}\right\}\ /.\ nominals$$

3.33125

Check that with these values, the mean is still on target

$$\delta check = \frac{8\,F}{d^4\,G}\ Nb\ Db^3\ /.\ nominals$$

5.

Calculate the best coefficient of variation

```
δhatBest =
 9 (sD / Db) ^2 + (sN / Nb) ^2 + (sF / F) ^2 + 16 (sd / d) ^2 +
    (sG / G) ^2 /. nominals /. stdevs
```

0.0187547

$s\delta Best = 5\sqrt{\delta hatBest}$

0.68474

```
tolδBest = 4 sδBest
```

```
2.73896
```

Saving on material: compare the lengths of wire between the original design and the best design.

```
Voriginal = N π D /. nominals
```

```
200 π
```

```
Vbest = Nb π Db
```

```
301.923
```

```
Vbest / Voriginal 100
```

```
48.0525
```

In sum:

With the nominal values, the standard deviation of the deflection is 0.825 mm compared to the robustified value of 0.685 mm.

The material used in the robustified design is 48% of that in the original design.